Quiz 3

1 Green's functions (10 points). Consider the inhomogeneous differential equation

$$(1+x^2)y'' = f(x)\,, \qquad \text{with} \qquad y(0) = 0 \quad \text{and} \quad y(1) = 0\,.$$

1. Write down the equation satisfied by the Green's function $G(x,z)$ for this problem.
2. Find the "jump condition" satisfied by $G'(x,z)$ at $x = z$.
3. The Green's function is

$$G(x,z) = \begin{cases} \dfrac{x(z-1)}{1+z^2} & \text{for} \quad x < z \\[2ex] \dfrac{z(x-1)}{1+z^2} & \text{for} \quad x > z \end{cases}$$

   Verify this satisfies the appropriate boundary conditions and conditions at $x = z$.

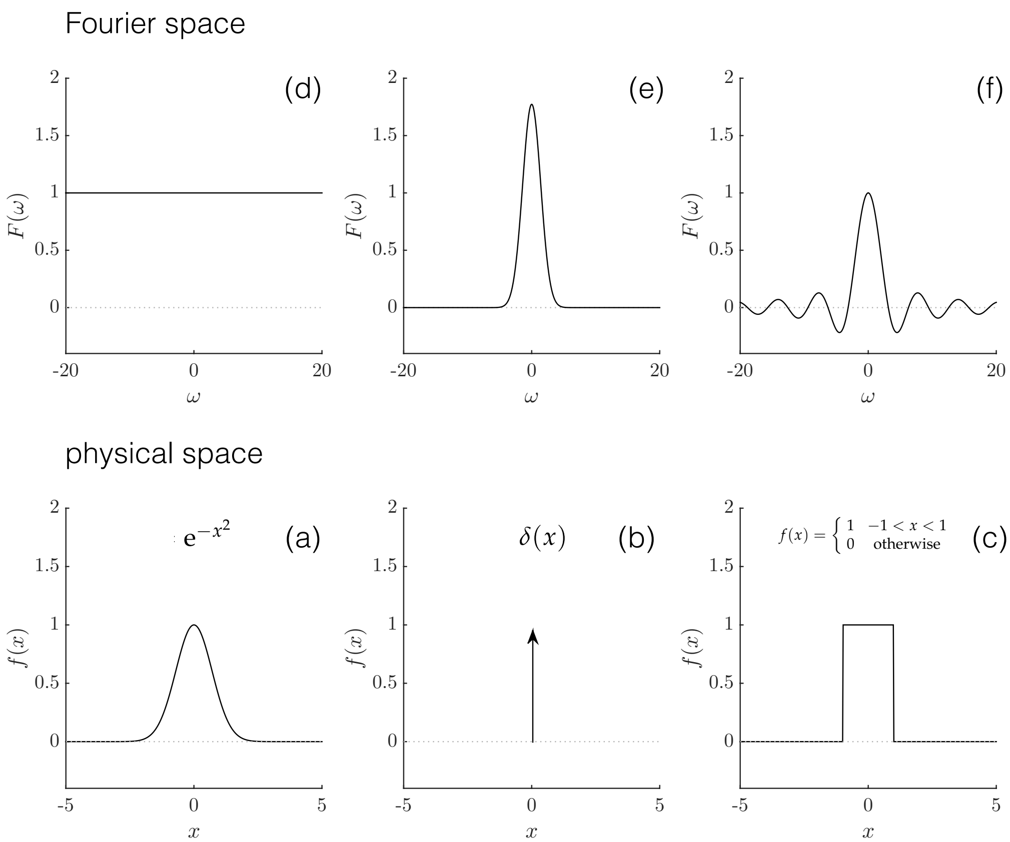
4. Write down the general solution for $y(x)$ in terms of two integrals. [*Hint: be very clear about which variable is the variable of integration as well as its range in each integral.*]

2 Fourier transforms (5 points). Three Fourier transforms are marked (d), (e), and (f) below. Match each of these transforms to the correct physical space function:

(a) $f(x) = \delta(x)$

(b) $f(x) = \mathrm{e}^{-x^2}$.

(c) $f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$



3 Multidimensional partial differential equations (10 points). Ponder for a moment the vibrations of a square drum modelled by the displacement of an elastic square membrane with corners at $(0,0)$, $(L,0)$, $(0,L)$ and (L,L). The displacement of the membrane is governed by the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) .$$

The membrane is held taut around its edge, so that

$$u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,L,t) = 0 .$$

We assume that, initially, the membrane has some finite displacement, but zero velocity, so that

$$u(x,y,0) = \phi(x,y) , \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,y,0) = 0 .$$

1. Separate variables by assuming that $u = S(x,y)g(t)$, propose a separation variable κ^2, and solve the t-equation in terms of c and κ. Make sure you account for the zero-velocity initial condition. What is the physical meaning of the product κc?

2. The solutions for $S(x,y)$ are

$$S(x,y) = A_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) ,$$

   where κ is found to be

$$\kappa_{nm} = \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2} ,$$

   and both n and m go from 1 to $+\infty$. Explain in a few lines how one would derive this result.

3. What are the lowest three frequencies of the elastic membrane?