Quiz 3 Solutions

1 Green's functions (10 points). Consider the inhomogeneous differential equation

$$(1+x^2)y'' = f(x)\,, \qquad \text{with} \qquad y(0)=0 \quad \text{and} \quad y(1)=0\,.$$

1. Write down the equation satisfied by the Green's function $G(x,z)$ for this problem.
2. Find the "jump condition" satisfied by $G'(x,z)$ at $x=z$.
3. The Green's function is

   $$G(x,z) = \begin{cases} \dfrac{x(z-1)}{1+z^2} & \text{for} \quad x<z \\[2ex] \dfrac{z(x-1)}{1+z^2} & \text{for} \quad x>z \end{cases}$$

   Verify this satisfies the appropriate boundary conditions and conditions at $x=z$.
4. Write down the general solution for $y(x)$ in terms of two integrals. [*Hint: be very clear about which variable is the variable of integration as well as its range in each integral.*]

Solution.

1. The Green's function satisfies

   $$(1+x^2)G'' = \delta(x-z)\,, \qquad \text{with} \qquad G(0)=0 \quad \text{and} \quad G(1)=0\,.$$

2. The jump condition is found by integrating the equation from 1 over a small neighborhood surrounding z. The easiest way is to divide by $1+x^2$. Then we have

   $$G'' = \frac{\delta(x-z)}{1+x^2}\,,$$

   and integration yields

   $$G'(z^+) - G'(z^-) = \frac{1}{1+z^2}\,.$$

   Another way is to integrate

   $$(1+x^2)G'' = \delta(x-z)\,.$$

If we do this we need to integrate by parts to find

$$\lim_{\epsilon\to 0}\int_{z-\epsilon}^{z+\epsilon}(1+x^2)G''\,\mathrm{d}x = \int_{z-\epsilon}^{z+\epsilon}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1+x^2)G'\right]-2xG''\,\mathrm{d}x = \left(1+z^2\right)\left[G'(z^+)-G'(z^-)\right].$$

This yields the same jump condition as above, since $\int_{z-\epsilon}^{z+\epsilon}\delta(x-z)\,\mathrm{d}x = 1$, and so

$$G'(z^+)-G'(z^-) = \frac{1}{1+z^2}.$$

3. At $x=0$ we use the Green's function valid for $x<z$, which is

$$G(0,z) = \frac{x(z-1)}{1+z^2}\bigg|_{x=0} = 0,$$

since the whole thing is multiplied by x. At $x=1$,

$$G(1,z) = \frac{z(x-1)}{1+z^2}\bigg|_{x=1} = 0,$$

since $x-1=0$ at $x=1$. At $x=z$, both halves of the Green's function are equal to

$$\frac{z(z-1)}{1+z^2}.$$

The derivative of the Green's function is

$$G'(x,z) = \begin{cases}\dfrac{z-1}{1+z^2} & \text{for} \quad x<z\\[2ex] \dfrac{z}{1+z^2} & \text{for} \quad x>z\end{cases}$$

Thus we find

$$G'(z^+)-G'(z^-) = \frac{z}{1+z^2}-\frac{z-1}{1+z^2} = \frac{1}{1+z^2},$$

and all the conditions are satisfied.

4. The general solution for $y(x)$ is

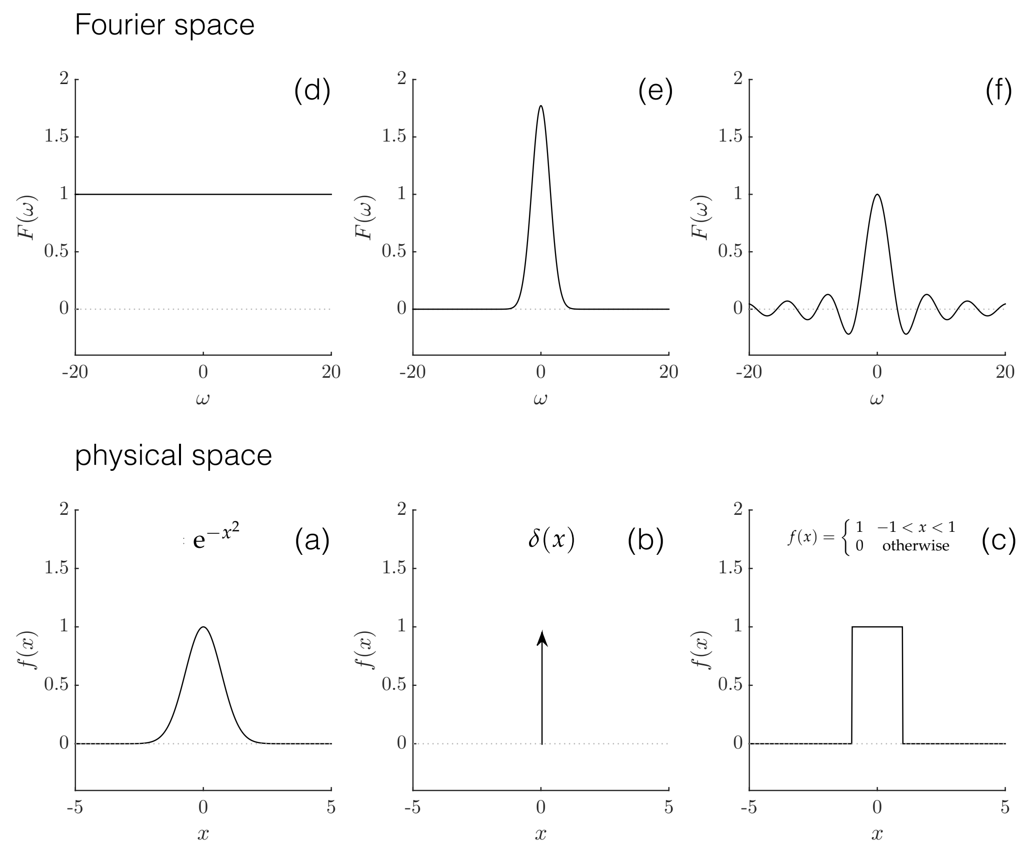
$$\begin{aligned} y(x) &= \int_0^1 f(z)G(x,z)\,\mathrm{d}z, && (1)\\ &= \underbrace{\int_0^x f(z)G(x>z,z)\,\mathrm{d}z}_{\text{use } G \text{ for } z<x} \;+\; \underbrace{\int_x^z f(z)G(x<z,z)\,\mathrm{d}z}_{\text{use } G \text{ for } z>x}, && (2)\\ &= (x-1)\int_0^x f(z)\frac{z}{1+z^2}\,\mathrm{d}z \;+\; x\int_x^1 f(z)\frac{z-1}{1+z^2}\,\mathrm{d}z. && (3)\end{aligned}$$

2 Fourier transforms (5 points). Three Fourier transforms are marked (d), (e), and (f) below. Match each of these transforms to the correct physical space function:

(a) $f(x) = \delta(x)$

(b) $f(x) = \mathrm{e}^{-x^2}$.

(c) $f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$



Solution. Note that the letters (a,b,c) on the figures **do not** *correspond to the letters in the problem write up. Unfortunately.*

The solution is

$$(d) \to \delta(x) \quad \text{(a)}\,, \tag{4}$$

$$(e) \to \mathrm{e}^{-x^2} \quad \text{(b)}\,, \tag{5}$$

$$(f) \to \text{square pulse} \quad \text{(c)}\,. \tag{6}$$

3 Multidimensional partial differential equations (10 points). Ponder for a moment the vibrations of a square drum modelled by the displacement of an elastic square membrane with corners at $(0,0)$, $(L,0)$, $(0,L)$ and (L,L). The displacement of the membrane is governed by the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

The membrane is held taut around its edge, so that

$$u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,L,t) = 0\,.$$

We assume that, initially, the membrane has some finite displacement, but zero velocity, so that

$$u(x,y,0) = \phi(x,y)\,, \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,y,0) = 0\,.$$

1. Separate variables by assuming that $u = S(x,y)g(t)$, propose a separation variable κ^2, and solve the t-equation in terms of c and κ. Make sure you account for the zero-velocity initial condition. What is the physical meaning of the product κc?

2. The solutions for $S(x,y)$ are

   $$S(x,y) = A_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)\,,$$

   where κ is found to be

   $$\kappa_{nm} = \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2}\,,$$

   and both n and m go from 1 to $+\infty$. Explain in a few lines how one would derive this result.

3. What are the lowest three frequencies of the elastic membrane?

Solution.

1. Substituting $u = S(x,y)g(t)$, dividing by $c^2 Sg$, and defining a separation constant κ^2 yields

   $$\frac{g''}{c^2 g} = \frac{\nabla^2 S}{S} = -\kappa^2\,.$$

   The t-equation is

   $$g'' + (c\kappa)^2 g = 0\,,$$

   and the general solution is

   $$g = A\sin(\kappa c t) + B\cos(\kappa c t)\,.$$

Since at $t = 0$ we have

$$\frac{\partial u}{\partial t} = 0\,,$$

This means that $A = 0$, and the solution to the t-equation is

$$g = B\cos(\kappa c t)\,.$$

The product κc is the oscillation frequency of the mode corresponding to κ. Note that κ has units of 1/length, and c has units of length/time, so that κc has units 1/time.

2. The result for S is found by forming the (x, y)-equation,

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \kappa S = 0\,,$$

separating variables, defining a new separation constant, and solving each equation for x and y separately. Both x and y-equations are eigenproblems which are each associate with an infinite sum of eigenvalues; this leads to a doubly-infinite sum of solutions for $S(x, y)$, each associated with a particular value of κ_{nm}.

3. Because the spatial modes are sines, the smallest values of n and m are $n = m = 1$; and the smallest values of n and m correspond to the smallest values of κ and therefore the smallest values of κc, the eigenfrequencies of the membrane. The lowest frequency is associated with $n = m = 1$, and the next two greater frequencies have $n = 1$, $m = 2$ and $n = 2$, $m = 1$. Thus the lowest three frequencies are

$$\kappa c \;=\; \left(\tfrac{c\pi\sqrt{2}}{L}\,,\; \tfrac{c\pi\sqrt{5}}{L}\,,\; \tfrac{c\pi\sqrt{5}}{L} \right).$$