Vector calculus in (almost) one page.

You should know the following operators and formulas. We use Cartesian coordinates $\mathbf{r} = \mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$. Write the vectors $\mathbf{a}$ and $\mathbf{b}$ as $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$; $f(\mathbf{x})$ is a scalar function of $\mathbf{x}$ and $\mathbf{u}(\mathbf{x}) = (u_1, u_2, u_3)$ is a vector function of $\mathbf{x}$.

Dot product: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$. The dot product of two vectors is a *scalar*.

Cross product: $\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$. The cross product of two vectors is a *vector*.

Gradient:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$$

The gradient of a scalar function is a *vector*.

Divergence:

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}.$$

The divergence of a vector function is a *scalar*.

Curl:

$$\nabla \times \mathbf{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right).$$

The curl of a vector function is a *vector*.

Differentials:

$$\mathrm{d}f = \frac{\partial f}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\,\mathrm{d}x_2 + \frac{\partial f}{\partial x_3}\,\mathrm{d}x_3.$$

Divergence theorem:

$$\int_V \nabla \cdot \mathbf{u}\,\mathrm{d}V = \int_S \mathbf{u} \cdot \mathbf{n}\,\mathrm{d}S,$$

where $\mathbf{n}$ is the unit vector oriented outward from the volume V.

Stokes' theorem:

$$\int_S \nabla \times \mathbf{u} \cdot \mathbf{n}\,\mathrm{d}S = \int_C \mathbf{u} \cdot \mathrm{d}\mathbf{l},$$

where C is any curve bounding the open surface S.

Show that given $\mathbf{a} = (2x, 3xy, 0)$, $\mathbf{b} = (2, x, 0)$ and $f = 3xy^2$, $\mathbf{a} \cdot \mathbf{b} = 4x + 3x^2 y$, $\nabla \cdot \mathbf{a} = 2 + 3x$, $\mathbf{a} \times \mathbf{b} = (0, 0, 2x^2 - 6xy)$, $\nabla \times \mathbf{a} = (0, 0, 3y)$, $\nabla f = (3y^2, 6xy, 0)$, $\nabla \times f =$ absurd.

Suffices and the summation convention: The right way to do vector calculus is using suffices. The vector $\mathbf{a}$ is written as a_i and repeated indices are summed over. The above operators become

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i; \qquad [\mathbf{a} \times \mathbf{b}]_i = \epsilon_{ijk} a_j b_k.$$

where $\epsilon_{ijk} = 1$ if (i, j, k) are an even permutation of $(1, 2, 3)$, -1 if they are an odd permutation and 0 otherwise.
The Kronecker delta is defined by $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Note that

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

Treating the gradient operator as the vector ∇, which is ∂_i in suffix notation, the other equations become

$$[\nabla f]_i = \frac{\partial f}{\partial x_i}; \qquad \nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i}; \qquad [\nabla \times \mathbf{u}]_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}; \qquad \mathrm{d}P = \frac{\partial P}{\partial x_i}\,\mathrm{d}x_i.$$

This is the only way to do tensors, which have two or more free suffices.

Show using suffices that $\nabla \times \nabla f = \mathbf{0}$, $\nabla \cdot (\nabla \times \mathbf{u}) = 0$, $\delta_{ii} = 3$ in 3 dimensions. **Compute using suffices** $\mathbf{a} \times (\mathbf{b} \times \mathbf{b})$, $\nabla \times (\nabla \times \mathbf{u})$.