MAE210A: Fluid Mechanics I Fall Quarter 2006
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Solution Homework 2 Exercise 3

Quite a few people had trouble with the concept of linearization. Here is a detailed explanation of what was asked in this problem.

The equations governing the flow are the compressible forms of the continuity equation and the Navier–Stokes equation. For simplicity, we'll make here the Stokes assumption $\lambda = -(2/3)\mu$. The text asked you to keep viscous terms. Nothing was said about body forces and you can't assume that p_0 is uniform if you introduce body forces. We therefore have

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{u}) &= 0, \\ \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) &= -\nabla p + \mu\nabla^2\mathbf{u} + \frac{\mu}{3}\nabla(\nabla\cdot\mathbf{u}). \end{aligned}$$

Let's consider a base state $(\mathbf{u}_0, p_0, \rho_0)$. This must be a solution of the previous equations. You were also given in the text that this reference state is at rest ($\mathbf{u}_0 = 0$) and that the pressure and density *of the base state* are uniform (p_0 and ρ_0 don't depend on position). Substituting these assumptions in the continuity equation for the base state gives $\partial\rho_0\partial t = 0$ so the density of the base state is a constant. Navier–Stokes is automatically satisfied. Note that if you had introduced a body force on the right hand side, you couldn't have p_0 uniform, but $\nabla p_0 = \rho_0 \mathbf{g}$.

Let's now introduce a perturbation to this reference solution of the equations of motion: $(\mathbf{u}', p', \rho')$. The fundamental assumption is that these quantities are small. Hence a term containing a product of two (or more) of them will be negligible with respect to terms that are linear in prime quantities. Introducing the decomposition $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$, $\rho = \rho_0 + \rho'$ and $p = p_0 + p'$ and substituting, we get for the continuity equation

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho'}{\partial t} + \nabla\cdot(\rho_0\mathbf{u}_0 + \rho_0\mathbf{u}' + \rho'\mathbf{u}_0 + \rho'\mathbf{u}') = 0.$$

As discussed above, we know that $\mathbf{u}_0 = 0$ and ρ_0 is a constant (independent of both space and time). Additionally $\rho'\mathbf{u}'$ are of second order in the perturbation quantities and hence negligible with respect to $\rho_0\mathbf{u}'$ for example. Applying these simplifications leads to

$$\frac{\partial \rho'}{\partial t} + \rho_0(\nabla\cdot\mathbf{u}') = 0. \tag{1}$$

Carrying out the same substitution and expansion in the Navier–Stokes equations gives

$$(\rho_0+\rho')\left(\frac{\partial \mathbf{u}_0}{\partial t}+\frac{\partial \mathbf{u}'}{\partial t}+(\mathbf{u}_0+\mathbf{u}')\cdot\nabla(\mathbf{u}_0+\mathbf{u}')\right)$$
$$=-\nabla p_0-\nabla p'+\mu\nabla^2\mathbf{u}_0+\frac{\mu}{3}\nabla(\nabla\cdot\mathbf{u}_0)+\mu\nabla^2\mathbf{u}'+\frac{\mu}{3}\nabla(\nabla\cdot\mathbf{u}').$$

Remember that $\mathbf{u}_0=0$ and $\nabla p_0=0$ (or $\nabla p_0=\rho_0\mathbf{g}$ if you decided to include body forces). Terms like $\rho'\partial\mathbf{u}'/\partial t$, $\rho_0(\mathbf{u}'\nabla)\mathbf{u}'$ are of second order in the perturbation quantities and will be neglected with respect to terms linear in the prime quantities. The Navier–Stokes equations hence become in linearized form

$$\rho_0\frac{\partial \mathbf{u}'}{\partial t}=-\nabla p'+\mu\nabla^2\mathbf{u}'+\frac{\mu}{3}\nabla(\nabla\cdot\mathbf{u}'). \quad (2)$$

Let's now think about what is exactly the problem we are trying to solve. The base state is known, so ρ_0 and p_0 are not considered as unknowns here. Our unknowns are p', ρ' and $\mathbf{u}'$, which is a total of 5 scalars. We have 4 equations. Hence we need another.

The quantities ρ' and p' are small variations of the base state density ρ_0 and pressure p_0. The transformation from (ρ_0,p_0) to $(\rho_0+\rho',p_0+p')$ can therefore be considered as adiabatic (no heat exchange) and reversible (small changes; technically, reversibility is incompatible with the presence of viscosity but this irreversible process can be neglected as a first approximation here). Therefore the entropy of is conserved. For a perfect gas, the entropy is proportional to $\log(p\rho^{-\gamma})$ (see the thermodynamics review in Kundu), and an isentropic transformation conserves s and therefore conserves $p\rho^{-\gamma}$. Therefore, $p_0\rho_0^{-\gamma}=(p_0+p')(\rho_0+\rho')^{-\gamma}$. This equation should now be linearized by performing a Taylor expansion of the right hand side.

$$(p_0+p')(\rho_0+\rho')^{-\gamma}=p_0\rho_0^{-\gamma}\left(1+\frac{p'}{p_0}\right)\left(1+\frac{\rho'}{\rho_0}\right)^{-\gamma}\sim p_0\rho_0^{-\gamma}\left(1+\frac{p'}{p_0}-\gamma\frac{\rho'}{\rho_0}\right).$$

This should be equal to $p_0\rho_0^{-\gamma}$ and therefore at leading order:

$$p'=\gamma\frac{p_0}{\rho_0}\rho'=\gamma RT_0\rho'=c^2\rho', \quad (3)$$

where $c^2=\gamma p_0/\rho_0=\gamma RT_0$ is the square of the speed of sound for a perfect gas. Another way of saying that is that the definition of the speed of sound is $c^2=(\partial p/\partial\rho)_s$. Since p' and ρ' are small variations of the base state quantities and the process is assumed to be isentropic, we can write $c^2\sim p'/\rho'$.

Equations (1),(2) and (3) are the linearized equations of motions that were asked for in this problem.