Final

This is a three-hour closed-book open-note exam. No calculators. Attempt all six questions.

1 (15 points) Compute the divergence and curl of the velocity field

$$\mathbf{u} = \left(xy, y, \frac{1+z^2}{1+t^2} \right).$$

What is the pathline for a particle released at $(1, 1, 1)$ at $t = 0$? Show that a particle released from the origin moves along a streamline.

2 (15 points) James Bond is served a martini in a conical glass: a layer with depth d_1 of vodka (density ρ_1) floating above a layer with depth d_2 of vermouth (density ρ_2). After the martini is stirred, does the pressure at the base of the glass go up or down? A SMERSH agent drops an exceedingly small grain of poison (density ρ_3, diameter l) which sinks slowly through the stirred cocktail. Calculate the terminal velocity of the grain. [The volume of a cone with base area a and height h is $V = ah/3$.]

3 (15 points) At a point $\mathbf{x}$ along a streamline, define the unit vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ where $\mathbf{t}$ is parallel to the streamline (tangent vector) and $\mathbf{n}$ and $\mathbf{b}$ are normal (respectively normal and binormal vectors). The corresponding coordinates are s, n and b. Show that the steady Euler equations with constant density and a conservative body force can be written in the form

$$u \frac{\partial}{\partial s}(u\mathbf{t}) = -\frac{1}{\rho}\nabla p - \nabla \Omega,$$

where $u = |\mathbf{u}|$. Using the relation $\partial \mathbf{t}/\partial s = \mathbf{n}/R$, deduce the three equations

$$\frac{\partial}{\partial s}\left[\tfrac{1}{2}u^2 + \frac{p}{\rho} + \Omega\right] = 0, \qquad \frac{u^2}{R} = -\frac{\partial}{\partial n}\left[\frac{p}{\rho} + \Omega\right], \qquad \frac{\partial}{\partial b}\left[\frac{p}{\rho} + \Omega\right] = 0.$$

Give the physical meaning of these three equations. What is R?

4 (15 points) A bubble of radius $R(t)$ is immersed in a fluid with constant density. Assuming that the flow in the fluid is radial and that the fluid is inviscid, find the velocity potential and show that the pressure in the fluid is given by

$$p = p_\infty + \rho \left(\frac{R^2 \ddot{R} + 2R\dot{R}^2}{r} - \frac{R^4 \dot{R}^2}{2r^2} \right).$$

Deduce the equation of motion for $R(t)$, (i) assuming that the pressure in the bubble is negligible and then (ii) assuming that the gas in the bubble is adiabatic with no gas diffusing through the boundary of the bubble.

5 (15 points) A circular pipe contains fluid with density ρ_1 and dynamic viscosity μ_1 in the region $0 < r < a$ and fluid with density ρ_2 and dynamic viscosity μ_2 in the region $a < r < b$. Compute the volume flux Q as a function of the pressure difference Δp over a length L. Then compute the shear stress at the boundary.

6 (15 points) The viscosity μ of a liquid can be measured by determining the time t it takes for a sphere of diameter d to settle slowly through a distance l in a vertical cylinder of diameter D containing the liquid. Assume that $t = f(l, d, D, \mu, \Delta\rho)$, where $\Delta\rho$ is the difference in densities between the sphere and the liquid. Use dimensional analysis to show how t is related to μ, and describe how such an apparatus might be used to measure viscosity.