Homework 2

Due Oct 17, 2006.

1 [Adapted from Acheson 1.7] Sketch the streamlines for the flow

$$u = \alpha x, \qquad v = -\alpha y, \qquad w = 0,$$

where α is a positive constant.

Let the concentration of some pollutant in the fluid be

$$c(x, y, t) = \beta x^2 y \mathrm{e}^{-\alpha t},$$

for $y > 0$, where β is a constant. Does the pollutant concentration for any particular fluid element change with time?

An alternative way of describing any flow is to specify the position $\mathbf{x}$ of each fluid element at time t in terms of the position $\mathbf{X}$ of that element at time $t = 0$. Find $\mathbf{x}(\mathbf{X}, t)$. Verify by direct calculation that

$$\left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}} = \mathbf{u}, \qquad \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{X}} = \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$$

in this particular case. Why are these results true in general? Find $c(X, Y, t)$.

2 Compute the element of the viscous stress in plane polar coordinates, i.e. t_r and t_θ in terms of u_r and u_θ.

3 Derive the linearized equations of motion for a viscous fluid starting from the compressible Navier–Stokes equations. Take a basic state at rest with uniform pressure and density; use the perfect gas law. [This means write $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$ and so on. You will have to think about thermodynamics.]

4 The equation for the gravitational potential Φ is Poisson's equation $\nabla^2\Phi = 4\pi G\rho$. Using $\mathbf{g} = -\nabla\Phi$, show that for a fluid at rest with spherical symmetry

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = -4\pi G\rho.$$

If a planet's density ρ_0 is constant and the pressure vanishes at the surface $r = a$, find the pressure at the center of the planet. What do you obtain for the Moon (mass 7.3×10^{22} kg, radius 1738 km)?