Midterm Solutions

1 (i) Continuity: $\nabla\cdot\mathbf{u} = \partial_x(Uy/h) = 0$: OK. Navier–Stokes: no velocity in the y- and z-directions. There may be hydrostatic balance in z but ignore it. In the x-direction

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = 0 = -\frac{\partial p}{\partial x} + \nu\nabla^2 u = -\frac{\partial p}{\partial x},$$

so this is a solution if there is no pressure gradient.

(ii) No pressure gradient in x; possibly hydrostatic gradient in z. Shear stress $\tau_{12} = \mu\mathrm{d}u/\mathrm{d}y = \mu U/h$.

(iii) Integral form: tendency and convective terms vanish. Pressure terms cancel. Only interesting component is x, and the surface stress terms much cancel:

$$0 = \int_S \tau_{1j}\,\mathrm{d}S_j = L(\tau_{12} - \tau_{12}) = 0$$

for any control volume of length L.

(iv) Mechanical energy equation (ignore body forces):

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(\tfrac{1}{2}\mathbf{u}^2) = u_i\frac{\partial\tau_{ij}}{\partial x_j}.$$

In this form all terms vanish. The right-hand side can also be written as

$$\frac{\partial}{\partial x_j}(u_i\tau_{ij}) - \phi.$$

These two terms are both

$$\frac{\partial}{\partial x_j}(u_i\tau_{ij}) = \frac{\partial}{\partial y}[(Uy/h)(\mu U/h)] = \mu U^2/h^2 = \phi.$$

Integral form: tendency and convective terms vanish. The surviving right-hand side terms are

$$\int_S u_i\tau_{ij}\,\mathrm{d}S_j = \int_V \phi\,\mathrm{d}V = \frac{\mu U^2 L d}{h^2}$$

where L is the length of the control volume and d is its height, since the integrands are constant.

2 (i) The boundary is a streamline from the form of ψ so the no-normal flow condition is satisfied there.

(ii) Calculate

$$u = -2A(t)\frac{y}{b^2}, \qquad v = 2A(t)\frac{x}{a^2}.$$

The vorticity is normal to the xy plane and has value

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2A(t)\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

i.e. it is constant in space. The rate of strain tensor is

$$e_{ij} = \begin{pmatrix} 0 & A(t)(a^{-2} - b^{-2}) \\ A(t)(a^{-2} - b^{-2}) & 0 \end{pmatrix}.$$

(iii) The circulation around the boundary of the ellipse can be written as $\int_S \omega \, \mathrm{d}A = A\omega$ since ω is constant. Using $A = \pi ab$ and the value of ω from (ii), we have

$$\Gamma = 2\pi abA(t)\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

(iv) Write down the two components of the Euler equation:

$$\begin{aligned} -\frac{2\dot{A}y}{b^2} - \frac{4A^2x}{(ab)^2} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x, \\ \frac{2\dot{A}x}{a^2} - \frac{4A^2y}{(ab)^2} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y. \end{aligned}$$

If $\dot{A} \neq 0$, it is impossible to find a solution to these equations without a body force (take ∂_y of the first minus ∂_x the second). If $\dot{A} = 0$, one can obtain a solution with no body force and

$$p = 2\rho A^2 \frac{x^2 + y^2}{(ab)^2}.$$

(v) In general, particle paths and streamlines are different, but in this flow they are the same even when the flow is unsteady. This is because the streamlines do not change over time, just the velocity of the flow along them.

3 The upper surface is at z and has velocity v. The exit is at 0 and has velocity v_0. By mass conservation we have

$$\pi h(z)^2 v = \pi h(0)^2 v_0.$$

Apply Bernoulli's equation on a streamline from the upper surface to the outlet. Assume that at the outlet the flow comes out as a constant-diameter jet so that the pressure there is atmospheric. Then we have

$$gz + \tfrac{1}{2}v^2 = \tfrac{1}{2}v_0^2.$$

Eliminate v_0 from these two equations:

$$gz + \tfrac{1}{2}v^2 = \tfrac{1}{2}v^2 \frac{h(z)^4}{h(0)^4}.$$

We require v to be constant in time, so this is not an ordinary differential equation but just an algebraic relation. The profile $h(z)$ is given by

$$h(z) = h(0)\left(1 + \frac{2gz}{v^2}\right)^{1/4}.$$