Solutions 0

1 Take the cross product with $\mathbf{u}$: $\mathbf{u}\times\mathbf{x} = \mathbf{u}\times\mathbf{v}$. Taking the cross product with $\mathbf{u}$ again leads to

$$(\mathbf{u}.\mathbf{x})\mathbf{u} - (\mathbf{u}.\mathbf{u})\mathbf{x} = (\mathbf{u}.\mathbf{v})\mathbf{u} - (\mathbf{u}.\mathbf{u})\mathbf{v}$$

Take the dot product with $\mathbf{u}$:

$$(\mathbf{u}.\mathbf{x})(1 + \mathbf{u}.\mathbf{u}) = \mathbf{u}.\mathbf{v} \tag{1}$$

Now eliminate $\mathbf{u}.\mathbf{x}$ between the two equations above:

$$\mathbf{x} = \mathbf{v} - \frac{\mathbf{u}.\mathbf{v}}{1 + \mathbf{u}.\mathbf{u}}$$

2 The extrema of a function on an unbounded domain are the points where its gradient vanishes. Here $\nabla f = (4x, 2y - 3)$ and f has one extremum at $(0, 3/2)$. We calculate $f_{xx}(3/2) > 0$ and $f_{yy}(3/2) > 0$, so $(0, 3/2)$ *is the minimum of f on the whole plane*.

However this point does not belong to the unit disk and hence the minimum of f over the disk is on the bounding circle. On this circle, write $x = \cos t, y = \sin t$ so that $F(t) = f(\cos t, \sin t) = \cos^2 t + 3(1 - \sin t)$. To find the extrema, take the derivative $F't = -\cos t(2\sin t + 3)$; its zeros are $t = \pm\pi/2$. To know the nature of the extrema, take the second derivative of F: $F''(t) = -2\cos 2t + 3\sin t$. Hence $F''(\pi/2) > 0$ and $F''(-\pi/2) < 0$. Therefore $t = \pi/2$ is the minimum on the unit disk and *the minimum on the unit disk is achieved at* $(0, 1)$.

3

- Remember that $\partial r/\partial x_i = \partial x_i/\partial r$. Then

$$\begin{aligned}
\nabla^2 r^{-n} &= \frac{\partial}{\partial x_i}(\frac{\partial r^n}{\partial x_i}) \\
&= \frac{\partial}{\partial x_i}(-nr^{-n-1}\frac{\partial r}{\partial x_i}) \\
&= \frac{\partial}{\partial x_i}(-nr^{-n-2}x_i) \\
&= n(n-2)r^{-n-4}x_i x_i + 3nr^{n-2} \\
&= n(n+1)r^{-n-2}.
\end{aligned}$$

- $\mathbf{\Omega}$ is a constant vector, so

$$[\nabla(\mathbf{\Omega}.\mathbf{x})]_j = \frac{\partial}{\partial x_j}(\Omega_k x_k) = \Omega_k \frac{\partial x_k}{\partial x_j} = \Omega_k \delta_{jk} = \Omega_j.$$

  Therefore $\nabla(\mathbf{\Omega}.\mathbf{x}) = \mathbf{\Omega}$.

- A,B, and C are constant so

$$\frac{\partial f}{\partial x_k} = A_{ij} x_i \delta jk + A_{ij} x_j \delta_{ik} + B_j \delta_{jk} = A_{ik} x_i + A_{kj} x_j + B_k.$$

  Hence $\nabla f = \mathbf{B} + (\mathbf{A} + \mathbf{A}^T)\mathbf{x}$. If A is symmetric, $\nabla f = 2\mathbf{A}\mathbf{x} + \mathbf{B}$ (this is identical to the differentiation rule for the case where x is a scalar)

4 Vector divergence:

$$\int_S p \mathrm{d}\mathbf{S} = \int_V \nabla p \mathrm{d}V \qquad \text{and} \nabla p = \frac{N^2 z}{g} \mathbf{e}_z.$$

Therefore $-\int_S p\mathrm{d}\mathbf{S} = -(\int_V z\mathrm{d}V) g^{-1} N^2 \mathbf{e}_z$ (pull the constant scalar and vectors out of the integral). The remaining integral is the weighted average of the z coordinate on the volume: this is the definition of z_c the centroid of the body. Hence

$$-\int_S p\mathrm{d}\mathbf{S} = -\frac{N^2}{g} z_c \mathbf{e}_z.$$

Note: if p is thought as the pressure field, the integral is the force on the body V. If one thinks of the body as being moved from a state of rest, the force tends to bring it back to its equilibrium position.

5 Since $\mathbf{u}$ is irrotational there exists a potential ϕ such that $\mathbf{u} = \nabla\phi$. We also need to assume that $\nabla\mathbf{u} = \nabla^2\phi = 0$. We obtain

$$\begin{aligned}
\int_V |\mathbf{u}|^2 \mathrm{d}V &= \int_V \nabla\phi.\nabla\phi \mathrm{d}V \\
&= \int_V [\nabla(\phi\nabla\phi) - \phi\nabla^2\phi] \mathrm{d}V \\
&= \int_V \nabla(\phi\nabla\phi) \mathrm{d}V \\
&= \int_S \phi(\nabla\phi.\mathrm{d}\mathbf{S}) \\
&= \int_S \phi \frac{\partial\phi}{\partial n} \mathrm{d}S.
\end{aligned}$$

This works if V is the domain enclosed by the surface. The case where V is unbounded should be handled very carefully. You need to use a volume bounded by your body and by a large sphere at infinity and check convergence. For example the integral diverges if $\mathbf{u}$ is constant.

6 The integral is a third-rank tensor (3 free indices), with 27 coefficients. It is also isotropic: given the shape of the domain and the symmetry of the integrand, the result doesn't change under rotation). There is only one third order isotropic tensor: ϵ_{ijk}, and hence the integral is proportional to this tensor $A_{ijk} = \kappa\epsilon_{ijk}$. Instead of 27 integrals we only need to compute 1 integral, for example $\int_V x_1x_2x_3\mathrm{d}V$. To compute this integral, use spherical coordinates $(x_1 = r\cos\theta,_2 = r\sin\theta\cos\phi, x_3 = r\sin\theta\sin\phi)$. The integral over the sphere $(0 \le r \le a, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi)$ becomes

$$\kappa = \underbrace{\int_0^a r^5 \mathrm{d}r}_{=\frac{a^6}{6}} \underbrace{\int_0^\pi \cos\theta\sin^3\theta \mathrm{d}\theta}_{=0} \underbrace{\int_0^{2\pi} \cos\phi\sin\phi\, \mathrm{d}\phi}_{=0} .$$

Therefore A is the zero third-rank tensor.

7 The components of the tensor in natural basis at the point $(2, 1, -1)$ are

$$\begin{pmatrix} 4-\sqrt{6} & 2 & -2 \\ 2 & 1-\sqrt{6} & -1 \\ -2 & -1 & 1-\sqrt{6} \end{pmatrix}.$$

The principle axes of the tensor are the eigenvectors. You need to compute the eigenvalues of the matrix and then solve for the eigenvectors. Notice that

$$T = \overbrace{\begin{pmatrix} 3 & 2 & -2 \\ 2 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix}}^{T_1} + (1-\sqrt{6})I$$

where I is the identity matrix. To find the eigenvalues of T we need only compute the eigenvalues of T_1 since $\lambda(T) = \lambda(T_1) + 1 - \sqrt{6}$. The characteristic polynomial of T_1 is $P_{T_1} = -(\lambda+1)^2(\lambda-5)$. The eigenvalues of T are therefore $(-\sqrt{6}, -\sqrt{6}, 6-\sqrt{6})$ and an associated orthonormal basis of eigenvectors is

$$V = \begin{pmatrix} \frac{1}{\sqrt{11}} & -\frac{4}{\sqrt{66}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{11}} & \frac{7}{\sqrt{66}} & \frac{1}{\sqrt{6}} \\ \frac{3}{\sqrt{11}} & -\frac{1}{\sqrt{66}} & -\frac{1}{\sqrt{6}} \end{pmatrix}.$$