Solutions Homework 1

1 Use suffices.

$$
\begin{aligned}
[\mathbf{u} \times (\nabla \times \mathbf{u})]_i &= \epsilon_{ijk} u_j \epsilon_{klm} \frac{\partial u_m}{\partial x_l} \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) u_j \frac{\partial u_m}{\partial x_l} \\
&= u_j \frac{\partial u_j}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j} \\
&= \frac{\partial}{\partial x_i}(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla) u_i
\end{aligned}
$$

Finally, $\mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{u}$

The left-hand side vanishes if $\mathbf{u}$ and $\nabla \times \mathbf{u}$ are parallel. This is clearly true if the flow is irrotational ($\nabla \times \mathbf{u} = 0$). But other velocity fields satisfy this property, for example $\mathbf{u} = (\sin z, \cos z, 0)$. For this field, $\mathbf{u} = \nabla \times \mathbf{u}$ and the left-hand side of the equation is zero, but $\mathbf{u}$ is not irrotational.

2 Use components.

- Divergence:

$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 3\alpha + 2\beta(x + y + z).$$

- Curl:

$$\nabla \times \mathbf{u} = -2\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

- Strain rate tensor: $e_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. This tensor is symmetric, so we only need to compute 6 components.

$$e = \begin{pmatrix} \alpha + 2\beta x & \alpha & \alpha \\ \alpha & \alpha + 2\beta y & \alpha \\ \alpha & \alpha & \alpha + 2\beta z \end{pmatrix}.$$

Note that $e_{ii} = \nabla\cdot\mathbf{u}$.

3 At $t = 2\pi$, the equations for the streamlines are

$$-\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}y}{x} = (x^2 + y^2 + 2\pi + 1)\,\mathrm{d}z.$$

Parameterize with s. Then we need to solve

$$\frac{\mathrm{d}x}{\mathrm{d}s} = -y, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = x, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = \frac{1}{x^2 + y^2 + 2\pi + 1}.$$

The solution to the first two of these equations that passes through the point (x_0, y_0, z_0) is

$$x = x_0 \cos s - y_0 \sin s, \qquad y = x_0 \sin s + y_0 \cos s.$$

Hence $x^2 + y^2 = x_0^2 + y_0^2$ which is independent of s, and the solution to the final equation is

$$z = \frac{s}{x_0^2 + y_0^2 + 2\pi + 1} + z_0.$$

Note that in the above equation, there are actually only 2 independent free parameters. We can for example set y_0 to zero by redefining s. (This only changes the origin on the streamline not the streamline itself).

The equations for particle paths and streamlines are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{x^2 + y^2 + t + 1}.$$

The solution to the first two of these equations passing through $(1, 0, 0)$ at $t = t_*$ is

$$x = \cos{(t - t_*)}, \qquad y = \sin{(t - t_*)}.$$

Once again $x^2 + y^2 = 1$ is a constant. The final equation then has solution

$$z = \log \frac{t + 2}{t_* + 2}.$$

The particle paths correspond to $t_* = 0$ so

$$x = \cos t, \qquad y = \sin t, \qquad z = \log{(1 + t/2)}.$$

for $0 \le t \le 2\pi$. For streaklines, $t = 2\pi$ and

$$x = \cos t_*, \qquad y = -\sin t_*, \qquad z = \log \frac{2\pi + 2}{t_* + 2}.$$

where $0 \le t_* \le 2\pi$.

4 The circle can be parameterized by

$$\begin{cases} x(\theta) = 1 + 2\cos\theta \\ y(\theta) = 6 + 2\cos\theta \end{cases}, \qquad 0 < \theta < 2\pi.$$

On the circle, the tangent unit vector is $\mathbf{e}_\theta = -\sin\theta\,\mathbf{e}_x + \cos\theta\,\mathbf{e}_y$. The circulation is hence

$$\begin{aligned}
\Gamma &= \int_C (u\mathbf{e}_x + v\mathbf{e}_y)\cdot\mathbf{e}_\theta R\,\mathrm{d}\theta \\
&= \int_0^{2\pi} [\cos\theta(2(1+2\cos\theta) - 3(6+2\sin\theta))) - \sin\theta(3(1+2\cos\theta) + 6 + 2\sin\theta)]2\,\mathrm{d}\theta \\
&= 2\int_0^{2\pi} (-16\cos\theta + 4\cos^2\theta + 9\sin\theta - 2\sin^2\theta - 30\sin\theta\cos\theta))\,\mathrm{d}\theta \\
&= 4\pi.
\end{aligned}$$