MAE210A: Fluid Mechanics I Fall Quarter 2006
http://maecourses.ucsd.edu/mae210a

Solutions Homework 2

1 The flow is two-dimensional since $w = 0$ and $\partial_z = 0$. The streamlines satisfy $\mathrm{d}x/\alpha x = \mathrm{d}y/(-\alpha y)$ which can be written as $xy = C$ on a streamline. Streamlines are hence the hyperbolae $y = C/x$. The flow goes away from the y-axis.

The material derivative is given by

$$\frac{\mathrm{D}c}{\mathrm{D}t} = \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = -\alpha\beta x^2 y \mathrm{e}^{-\alpha t} + \alpha x(2\beta x y \mathrm{e}^{-\alpha t}) - \alpha y(\beta x^2 \mathrm{e}^{-\alpha t}) = 0;$$

c doesn't change in time following a fluid element.

At time t, the velocity of the particle that was at $\mathbf{X}$ at $t = 0$ is $\mathbf{u}(\mathbf{x}(\mathbf{X}, t), t)$, therefore for this particle

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x, \quad x(t=0) = X, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\alpha y, \qquad y(t=0) = Y.$$

This system can be solved to give $\mathbf{x}(\mathbf{X}, t) = (X\mathrm{e}^{\alpha t}, Y\mathrm{e}^{-\alpha t})$. Hence the Lagrangian derivatives taken at fixed $\mathbf{X}$ are

$$\left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}} = \begin{pmatrix} \alpha X e^{\alpha t} \\ -\alpha Y e^{-\alpha t} \end{pmatrix} = \begin{pmatrix} \alpha x \\ -\alpha y \end{pmatrix} = \mathbf{u}$$

and

$$\left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{X}} = \left(\frac{\partial}{\partial t}\begin{pmatrix} \alpha X e^{\alpha t} \\ -\alpha Y e^{-\alpha t} \end{pmatrix}\right)_{\mathbf{X}} = \alpha^2 \mathbf{x}.$$

The Eulerian acceleration is

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = (\mathbf{u}\cdot\nabla)\mathbf{u} = \left(\alpha x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y}\right)\mathbf{u} = \alpha^2\mathbf{x} = \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{X}}.$$

These results are true in general since they are respectively the definitions of velocity and acceleration written inthe Eulerian and Lagrangian formulations. Substituting for x and y in $c(x, y, t)$ we get

$$c(X, Y, t) = \beta X^2 Y$$

which is of course independent of t.

2 For a general Newtonian fluid, $\tau_{ij} = 2\mu e_{ij} + \lambda(\nabla\cdot\mathbf{u})\delta_{ij}$. Remember that $\partial_\theta \mathbf{e}_r = \mathbf{e}_\theta$, $\partial_\theta \mathbf{e} = -\mathbf{e}_r$ and $\partial_r \mathbf{e}_r = \partial_r \mathbf{e}_\theta = 0$. By definition,

$$\begin{aligned}
e_{rr} &= ((\mathbf{e}_r\cdot\nabla)(u))\cdot\mathbf{e}_r = \left(\frac{\partial}{\partial r}(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta)\right)\cdot\mathbf{e}_r = \frac{\partial u_r}{\partial r},\\
e_{\theta\theta} &= ((\mathbf{e}_\theta\cdot\nabla)\mathbf{u})\cdot\mathbf{e}_\theta = \left(\frac{1}{r}\frac{\partial}{\partial\theta}(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta)\right)\cdot\mathbf{e}_\theta\\
&= \left[\frac{1}{r}(\frac{\partial u_r}{\partial\theta}\mathbf{e}_r + u_r\mathbf{e}_\theta + \frac{\partial u_\theta}{\partial\theta}\mathbf{e}_\theta - u_\theta\mathbf{e}_r\right]\cdot\mathbf{e}_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta},\\
e_{r\theta} &= \frac{1}{2}[(\mathbf{e}_\theta\cdot\nabla)\mathbf{u})\mathbf{e}_r + (\mathbf{e}_r\cdot\nabla)\mathbf{u})\cdot\mathbf{e}_\theta]\\
&= \frac{1}{2}\left[\frac{1}{r}\frac{\partial}{\partial\theta}(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta)\cdot\mathbf{e}_r + \frac{\partial}{\partial r}(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta)\cdot\mathbf{e}_\theta\right]\\
&= \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right] = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right]
\end{aligned}$$

The isotropic term $\delta_i j$ only gives terms in the t_{rr} and $t_{\theta\theta}$ components, and in polar coordinates $\nabla\cdot\mathbf{u} = \frac{1}{r}\partial(ru_r)/\partial r + \frac{1}{r}\partial u_\theta/\partial\theta$. Finally,

$$\begin{aligned}
t_{rr} &= 2\mu\frac{\partial u_r}{\partial r} + \lambda\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right),\\
t_{r\theta} &= t_{\theta r} = \mu\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta} + r\frac{\partial}{\partial r}(\frac{u_\theta}{r})\right],\\
t_{\theta\theta} &= 2\mu\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right) + \lambda\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right).
\end{aligned}$$

3 Start from Navier–Stokes with no body forces and the continuity equation:

$$\begin{aligned}
\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} &= -\nabla p + \mu\nabla^2\mathbf{u} + \frac{\mu}{3}\nabla\cdot(\nabla\cdot\mathbf{u}),\\
\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} &= 0.
\end{aligned}$$

Decompose according to $\mathbf{u} = \mathbf{0} + \mathbf{u}'$, $\rho = \rho_0 + \rho'$ and $p = p_0 + p'$ with p_0 and ρ_0 uniform and constant. The variables p' and ρ' are small departures about the state of rest, so that the transformation can be considered adiabatic and reversible (isentropic) so $p\rho^{-\gamma}$ is a constant and $p'/p_0 = \gamma\rho'/\rho_0$. Using the perfect gas law, this can be written as $p' = \gamma RT_0\rho'$.

Substitute the decomposition into the dynamical equations and keep only the terms linear in the prime quantities. The linearized compressible equations become

$$\begin{aligned}
\rho\frac{\partial\mathbf{u}'}{\partial t} &= -\nabla p' + \mu\nabla^2\mathbf{u}' + \frac{\mu}{3}\nabla\cdot(\nabla\cdot\mathbf{u}'),\\
\frac{\partial\rho'}{\partial t} + \rho_0\nabla\cdot\mathbf{u}' &= 0,\\
p' = \gamma RT_0\rho'.
\end{aligned}$$

4 For a fluid at rest, the Navier–Stokes equations become $0 = \rho \mathbf{g} - \nabla p$ (the hydrostatic relation). Taking the divergence of this equation gives

$$\nabla \cdot \left(\frac{1}{\rho} \nabla p \right) = \nabla \cdot \mathbf{g} = -\nabla^2 \Phi = -4\pi G \rho.$$

With spherical symmetry, p only depends on r, the distance from the origin. Then

$$\nabla p = \frac{\partial p}{\partial r} \mathbf{e}_r, \qquad \nabla \cdot \left(\frac{1}{\rho} \nabla p \right) = \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{1}{\rho} \frac{\mathrm{d}p}{\mathrm{d}r} \right).$$

This gives the equation

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{r^2}{\rho} \frac{\mathrm{d}p}{\mathrm{d}r} \right) = -4\pi G \rho.$$

If $\rho = \rho_0$ is constant, we integrate and obtain

$$\frac{r^2}{\rho} \frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{4}{3} \pi \rho_0^2 G r^3 + c.$$

For the pressure gradient to be finite at $r = 0$, c must be zero. Dividing by r^2 and integrating one more time with respect to r yields

$$p = p_s - \frac{2}{3} \pi \rho_0^2 G (r^2 - a^2)$$

with p_s and a respectively the surface pressure and the radius of the planet. Assuming $p_s = 0$ (vacuum at the surface), the pressure at the center is

$$p_c = \frac{2}{3} \pi \rho_0^2 G a^2.$$

With the values $\rho_0 = 3.32 \times 10^3$ kg m^{-3}, $G = 6.67 \times 10^{-11}$ kg^{-1} m^3 s^{-2}, we obtain $p_c = 4.65 \times 10^9$ or 4.59×10^4 atm.