Solutions Homework 3

1 If we don't neglect the movement of the free surface of the tank, the velocity at the surface is $\mathbf{u} = \dot{h}\mathbf{e}_z$ and the potential in the tank can be taken to be $\phi = \dot{h}z$. If at the exit of the pipe, $\mathbf{u} = U\mathbf{e}_x$ and the potential is $\phi = Ux$ as in class, irrotational Bernoulli applied at the free surface and at the exit of the pipe gives

$$\ddot{h}h + \frac{1}{2}\dot{h}^2 + gh + \frac{p_0}{\rho} = \dot{U}L + \frac{1}{2}U^2 + \frac{p_0}{\rho}.$$

The velocities U and $\dot{h}$ can be related using mass conservation by $U = -(A/a)\dot{h}$ ($\dot{h}$ is negative and U is positive).

Substituting into Bernoulli's equation, we get an equation for h:

$$\left(h + \frac{A}{a}L\right)\ddot{h} + \frac{1}{2}\left(1 - \frac{A^2}{a^2}\right)\dot{h}^2 + gh = 0.$$

The dimensional quantities in the problem are h, h_0, L, g, A, a and t. There are therefore a priori 5 independent nondimensional group. Write $\beta = A/a$ and denote the scales for time and h by τ and H respectively so that $h = Hz$. We obtain the equation

$$\left(z + \frac{\beta L}{H}\right)\ddot{z} + \frac{1}{2}(1 - \beta^2)\dot{z}^2 + \frac{g\tau^2}{H}z = 0.$$

This leads us to pick $H = L/\beta$ and $\tau = \sqrt{gH}$. This we can write down the equation with only the parameter β as

$$\ddot{z}(\beta^{-2}z + 1) + \frac{1}{2}(\beta^{-2} - 1)\dot{z}^2 + z = 0$$

with initial condition $z(0) = z_0 \equiv h_0 a/LA$.

This equation can be solved numerically, e.g. using MATLAB. Results are shown below. For large β, as in class, we can drop the β^{-2} terms: this gives the dashed curves which are a good approximation. The hyperbolic tangent solution from class corresponds in addition to replacing the z term by z_0: these are the dot-dash curves, which are not such a good approximation. Note that the actual solution oscillates forever; the curves below have been truncated when $h = 0$.

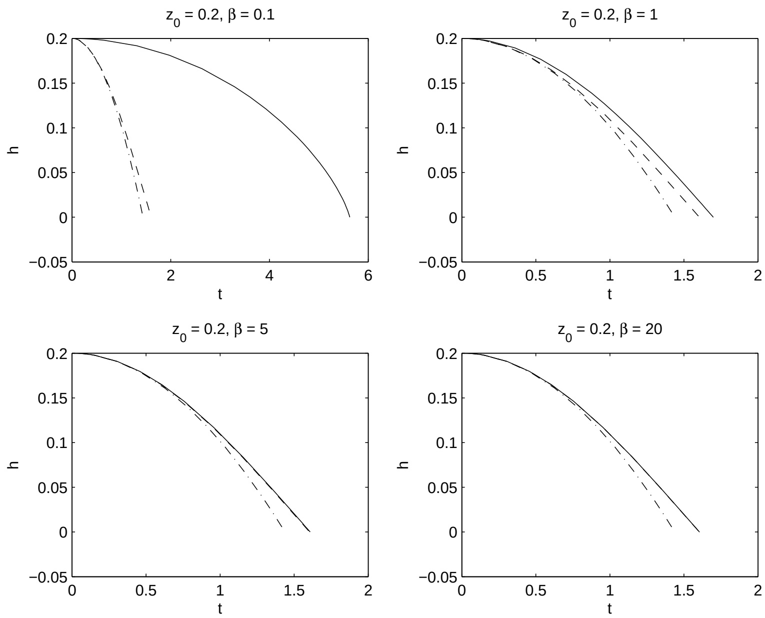

Figure 1: $h(z)$ for $z_0 = 0.2$.

2 Assume incompressible inviscid flow. There are two possibilities: work either in the lab frame where the fluid has velocity $\mathbf{u} = \Omega r \mathbf{e}_\theta$, or in the rotating frame where the fluid is at rest but you need to add a centrifugal body force: $\mathbf{f} = \Omega^2 r \mathbf{e}_r$.

In the rotating frame, the fluid is at rest so $\mathbf{u} = 0$. The forces acting to a fluid particle are gravity, pressure and centrifugal force. Projecting the Euler equation along the radial, azimuthal and vertical axes gives

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial r} + \rho \Omega^2 r, \\ 0 &= -\frac{\partial p}{\partial \theta}, \\ 0 &= -\frac{\partial p}{\partial z} - \rho g. \end{aligned}$$

Integrating this system gives $p = p_0 + \frac{1}{2}\rho\Omega^2 r^2 - \rho g z$.

Could you use Bernoulli? In the rotating frame, yes, because the body force is conservative and comes from a potential $\mathbf{f} = -\nabla F$ with $F = -\frac{1}{2}\Omega^2 r^2$. You can use irrotational Bernoulli and therefore the Bernoulli function $B = p/\rho - half\Omega^2 r^2 + \rho g z$ is a constant *everywhere* in the fluid.

If you stay in the lab frame, the flow is not irrotational anymore (solid body rotation) but steady. Steady Bernoulli would give you the same expression for

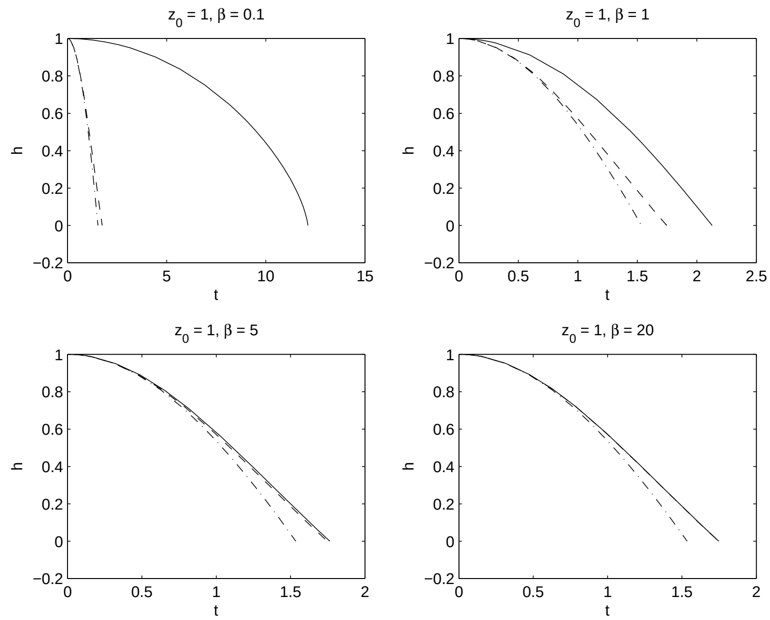

Figure 2: $h(z)$ for $z_0 = 1$.

the Bernoulli function B but this time this function is only a constant along streamlines (circles around the vertical axis) and vorticity lines (vertical lines). Therefore B does not depend on z and θ. But B is a function of r, and cannot be used to find the pressure everywhere.

The equation for the free surface is $z = \zeta(r)$. At the free surface, the boundary condition gives $p = p_a$. Therefore

$$p_a = p_0 + \frac{1}{2}\rho\Omega^2 r^2 - \rho g z.$$

Solving for ζ gives

$$\zeta = \frac{\Omega^2 r^2}{2g} + \zeta_0,$$

where ζ_0 is a constant.

3 Assume irrotational, inviscid and isothermal flow, and perfect gas. Then $\mathbf{u} = \nabla\phi$ and $p = \rho R T_0$. The continuity equation can be written as

$$\frac{\mathrm{D}\log\rho}{\mathrm{D}t} = \frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla\cdot\mathbf{u} = -\nabla^2\phi.$$

Unsteady Bernoulli gives

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + RT_0\log\rho = 0,$$

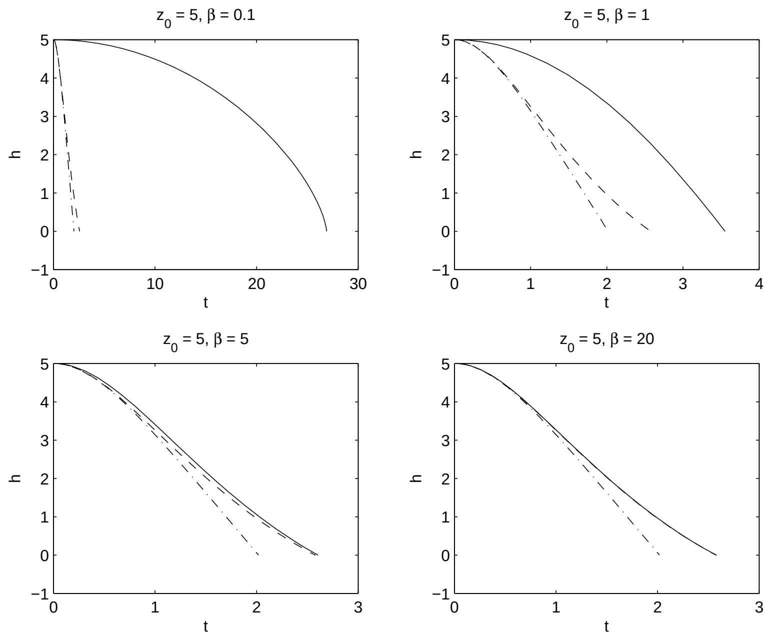

Figure 3: $h(z)$ for $z_0 = 5$.

since $\int \mathrm{d}p/\rho = RT_0 \int \mathrm{d}\rho/\rho = RT_0 \log \rho$. The constant in the unsteady Bernoulli equation can be set to 0 without loss of generality.

Substituting for $\log(\rho)$ in the continuity equation, we get an equation for ϕ only

$$RT_0 \nabla^2 \phi = \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2\right)$$

which can be expanded if necessary into

$$RT_0 \nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}(\nabla\phi)^2 + \frac{1}{2}\nabla\phi\cdot\nabla((\nabla\phi)^2).$$

If $\phi = \Gamma\theta/2\pi$, the velocity is

$$\mathbf{u} = \nabla\phi = \frac{\Gamma}{2\pi r}\mathbf{e}_\theta.$$

Substituting this into the ϕ equation shows that Γ must be constant in time. Substituting into the Bernoulli equation gives

$$\frac{\Gamma^2}{8\pi^2 r^2} + RT_0 \log p = C.$$

Applying the boundary condition at infinity gives

$$p = p_\infty e^{-\Gamma^2/(8\pi^2 R T_0 r^2)}.$$

For this isothermal compressible vortex, the pressure is zero at the center.

4 Start from the compressible form of the Euler equations in suffix notation

$$\frac{\partial u_k}{\partial t} + \frac{\partial}{\partial x_k}(\frac{1}{2}u_l u_l) + \epsilon_{klm}\omega_l u_m = -\frac{1}{\rho}\frac{\partial p}{\partial x_k} + \frac{\partial \Omega}{\partial x_k},$$

where Ω is the potential for the conservative body force. The gradient terms disappear when we take the curl, but the pressure term doesn't since ρ is no longer constant. As shown in lectures,

$$\begin{aligned}
[\nabla \times (\omega \times \mathbf{u})]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}(\epsilon_{klm}\omega_l u_m) \\
&= \frac{\partial}{\partial x_j}(\omega_i u_j - \omega_j u_i) \\
&= (\mathbf{u}\cdot\nabla)\omega_i + (\nabla\cdot\mathbf{u})\omega_i - (\omega\cdot\nabla)u_i.
\end{aligned}$$

For the pressure term,

$$\begin{aligned}
\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_k}\right) &= -\frac{\epsilon_{ijk}}{\rho^2}\frac{\partial \rho}{\partial x_j}\frac{\partial p}{\partial x_k} + \epsilon_{ijk}\rho\frac{\partial^2 p}{\partial x_j \partial x_k} \\
&= -\left(\frac{\nabla\rho \times \nabla p}{\rho^2}\right)_i,
\end{aligned}$$

since the last term cancels by symmetry. Substituting in the original equation gives in vector form

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = (\omega\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u})\omega + \frac{\nabla\rho \times \nabla p}{\rho^2}.$$

By the product rule

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\omega}{\rho} = \frac{1}{\rho}\frac{\mathrm{D}\omega}{\mathrm{D}t} - \frac{\omega}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}.$$

but the continuity equation gives $\mathrm{D}\rho/\mathrm{D}t\rho = -\rho(\nabla\cdot\mathbf{u})$. Combining this with the previous result gives

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\omega}{\rho} = \left(\frac{\omega}{\rho}\cdot\nabla\right)\mathbf{u} + \frac{\nabla\rho \times \nabla p}{\rho^3}.$$

The second term $\rho^{-2}\nabla\rho \times \nabla p$ creates vorticity when isopycnals and isobars are not aligned: it . It is the *baroclinic torque* or *rate of vorticity generation due to the baroclinicity of the flow*. For barotropic flow, p is a function of ρ only and the two gradients are aligned so the torque vanishes. We also see that in the absence of this term, ω/ρ is transported like a line element.