Solutions Homework 4

1 A is real so $\psi = 2Axy$. Streamlines are lines of constant $\psi = xy$, i.e. hyperbolae. The major and minor axes are the coordinate axes. The velocity field is $2A(x, -y)$, so if $A > 0$ the flow goes away from the y axis, and if $A < 0$ it goes away from the x axis. We have $|\mathbf{u}| = 2|A|\sqrt{x^2 + y^2} = 2|A|r$, so the speed is everywhere proportional to the distance from the origin.

2 Assume steady, incompressible, inviscid and irrotational flow. Consider the problem of the full cylinder with incoming flow at infinity $\mathbf{u} = U\mathbf{e}_x$. We have seen in class that $y = 0$ is a streamline for that flow. Therefore the flow above the x-axis is not modified if $y = 0$ is replaced by a solid boundary (here the ground), which is the configuration of a "Quonset hut". We therefore know the velocity potential in the fluid is $\phi = U\cos\theta(r + a^2/r)$, where a is the radius of the cylinder. The velocity on the boundary of the cylinder is $\mathbf{u} = -2U\sin\theta\mathbf{e}_\theta$. Applying irrotational Bernoulli between a point at infinity and a point on the surface of the cylinder leads to

$$p = p_\infty + \frac{1}{2}\rho_\infty U_\infty^2(1 - 4\sin^2\theta)$$

It is given that the pressure inside the cylinder is p_∞. Therefore the pressure force on the cylinder is $\mathbf{F} = \int(p_\infty - p)\mathbf{e}_r a\mathrm{d}\theta$. Projecting along the x-axis, we find $F_x = 0$ by symmetry. Along the y-axis,

$$F_y = a\int_0^\pi (p_\infty - p)\sin\theta\,\mathrm{d}\theta = -\tfrac{1}{2}\rho_\infty U_\infty^2 a\int_0^\pi (1 - 4\sin^2\theta)\sin\theta\,\mathrm{d}\theta = \frac{5}{3}\rho_\infty U_\infty^2 a.$$

This force is directing upward (the pressure outside is less than the pressure inside). Numerically, $a = 3$ m, $U_\infty = 40$ m/s and $\rho_\infty = 1.23$ kg/m^3. The force per unit depth is therefore $F_y = 9.84 \times 10^3$ N/m.

3 The moment about the origin of the pressure force $\mathrm{d}\mathbf{f} = -p\mathbf{n}\mathrm{d}l$ is $\mathrm{d}T = x\mathrm{d}f_y - y\mathrm{d}f_x$. Since $\mathbf{n}\mathrm{d}l = (\mathrm{d}y, -\mathrm{d}x)$, we have$\mathrm{d}T = p(x\mathrm{d}x + y\mathrm{d}y) = \mathrm{Re}(pz\mathrm{d}\bar{z})$. Bernoulli gives

$$p = p_\infty + \frac{1}{2}\rho U_\infty^2 - \frac{1}{2}\rho\frac{\mathrm{d}w}{\mathrm{d}z}\overline{\frac{\mathrm{d}w}{\mathrm{d}z}}.$$

Integrating over the whole boundary gives

$$T = \mathrm{Re}\left[\oint_C \left(p_\infty + \frac{1}{2}\rho U_\infty^2 - \frac{1}{2}\rho\frac{\mathrm{d}w}{\mathrm{d}z}\overline{\frac{\mathrm{d}w}{\mathrm{d}z}}\right) z\,\mathrm{d}\bar{z}\right].$$

The quantity $p_\infty + \frac{1}{2}\rho U_\infty^2$ is constant and can therefore be taken out of the integral. Furthermore, $\mathrm{Re}\left(\oint_C z\mathrm{d}\bar{z}\right) = \frac{1}{2}\oint_C (z\mathrm{d}\bar{z} + \bar{z}\mathrm{d}z) = \frac{1}{2}\oint_C \mathrm{d}(z\bar{z}) = 0$ on a closed contour.

The boundary is rigid so the no-normal flow condition is satisfied. That means that the boundary is a streamline. Along the boundary $\mathrm{d}\psi = \mathrm{Im}(\mathrm{d}w) = 0$ and hence $(\mathrm{d}w/\mathrm{d}z)\mathrm{d}z$ is a real number on the boundary. This can be rewritten as $(\mathrm{d}w/\mathrm{d}z)\mathrm{d}z = \overline{(\mathrm{d}w/\mathrm{d}z)\mathrm{d}z} = \overline{(\mathrm{d}w/\mathrm{d}z)}\mathrm{d}\bar{z}$. Substituting in into the expression for the moment, we obtain

$$T = \mathrm{Re}\left(-\frac{\rho}{2}\oint_C z\frac{\mathrm{d}w}{\mathrm{d}z}\mathrm{d}z\right).$$

4 The velocity potential of a dipole in two dimensions is

$$\phi = \frac{\mathbf{D}\cdot\mathbf{x}}{2\pi r^2}.$$

The corresponding velocity field is

$$u_i = \frac{\delta_{ij}r^2 - 2x_ix_j}{2\pi r^4}D_j.$$

We have one ipole at $(0, a)$ so that $\mathbf{x} = (x, y - a)$ and we take another with strength $\mathbf{D}'$ at $(0, -a)$ so that $\mathbf{x}' = (x, y + a)$. On the wall $y = 0$ and $r = r' = \sqrt{x^2 + a^2}$. Hence the vertical velocity is

$$v = \frac{2a(D_x - D'_x) + (2r^2 - 2a^2)(D_y + D'_y)}{2\pi r^4}.$$

For this to vanish, we take $\mathbf{D}' = (D_x, -D_y)$, i.e. the mirror image. Then

$$u = \frac{(2r^2 - 4x^2)D_x + 4axD_y}{2\pi r^4}.$$

The force on the plate is $-\int (p - p_\infty)\,\mathrm{d}\mathbf{S}$, where $\mathbf{n}$ points from the wall into the fluid, i.e. $\mathbf{n} = (0, 1)$. Bernoulli gives $p_\infty = p + \frac{1}{2}\rho u^2$ on the wall. Hence the force is normal to the wall with

$$F = \tfrac{1}{2}\rho\int_{-\infty}^{\infty}\left(\frac{(2r^2 - 4x^2)D_x + 4axD_y}{2\pi r^4}\right)^2 \mathrm{d}x.$$

The quantity F is positive so the force on the wall is positive, i.e. the force is up. Making the change of variable $x \to ax$ and writing $\mathbf{D} = D(\cos\alpha, \sin\alpha)$ gives

$$F = \frac{\rho D^2}{2\pi^2 a^3}\int_{-\infty}^{\infty}\frac{[(1 - x^2)\cos\alpha + 2x\sin\alpha]^2}{(1 + x^2)^4}\,\mathrm{d}x.$$

The integral can be shown to be equal to $\pi/4$ so $F = \rho D^2/8\pi^2 a^3$.

How can we calculate the integral? The easiest way is using contour integration. The integrand has a fourth-order pole at $z = \mathrm{i}$, so expand using $z = \mathrm{i} + \epsilon$:

$$\frac{1}{16\epsilon^4}[(2 - 2\mathrm{i}\epsilon - \epsilon^2)c + 2(\mathrm{i} + \epsilon)s]^2[1 - \tfrac{1}{2}\mathrm{i}\epsilon]^{-4} = \cdots - \frac{\mathrm{i}}{8\epsilon} + \cdots,$$

writing c and s for the cosine and sine terms. The residue theorem now gives $2\pi\mathrm{i} \times (-\mathrm{i}/8) = \pi/4$.

Another way is to make the substitution $x = \tan\theta$ and transform everything into linear trigonometric terms.