Solutions Homework 5

1 Gravity pulls the fluid downward, while viscous effects pull the fluid upward close to the walls. In the lab frame, the flow is purely vertical and fully developed so $\mathbf{u} = u(r)\mathbf{e}_z$. Assuming steady flow and no pressure gradient on the vertical, the vertical component of Navier Stokes is

$$0 = -g + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Solving for u by integrating twice in r gives $u(r) = gr^2/(4\nu) + A\log r + B$ with A and B constants to be determined. u must remain finite at $r = 0$ so $A = 0$. The wall is moving at velocity U upward, so $u(a) = U$ with a the radius of the pipe. Finally,

$$u(r) = \frac{g}{4\nu}(r^2 - a^2) + U$$

(Note that in the frame moving with the pipe wall, the flow is identical to the case seen in class for the pipe flow with an imposed pressure gradient: gravity plays the role of the pressure gradient). The volume flux through any horizontal section of the pipe is

$$Q = \int_0^a \int_0^{2\pi} u(r) r \,\mathrm{d}r\mathrm{d}\theta = 2\pi\left[\frac{g}{4\nu}\left(\frac{r^4}{4} - \frac{a^2r^2}{2}\right) + \frac{Ur^2}{2}\right]_0^a = \pi a^2\left(U - \frac{ga^2}{8\nu}\right).$$

Two effects are in competition: for small U, gravity is dominant and the net flow is downward. For large U, the viscous entrainment of the flow near the wall dominates gravity and the net flow is upward. There exists a critical value of U for which there is no net flow, namely $U_c = ga^2/(8\nu)$.

2 The flow is of the form $\mathbf{u} = u_\theta(r,t)\mathbf{e}_\theta$. The boundary condition at the cylinder wall is $u_\theta(r = a, t) = \Omega a \cos(\omega t)$. Ignoring transients $u(r,t)$ takes the form $u_\theta = \mathrm{Re}(\phi(r)e^{\mathrm{i}\omega t})$. The azimuthal component of Navier–Stokes is

$$\frac{\partial u_\theta}{\partial t} = \nu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right).$$

The velocity field and pressure gradient don't depend on θ (if $\partial p/\partial\theta$ is nonzero, the pressure is multivalued).

Substituting for u_θ gives an equation for ϕ:

$$\mathrm{i}\omega\phi = \nu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\phi)\right)$$

which can be rewritten as

$$r^2\phi'' + r\phi' - (\frac{\mathrm{i}\omega r^2}{\nu} + 1)\phi = 0, \qquad \phi(a) = \Omega a$$

with ϕ finite everywhere in the domain.

The change of variable $\eta = r(1+\mathrm{i})\sqrt{\omega/2\nu} = r\delta^{-1}(1+\mathrm{i})$ with $f(\eta) = \phi(r)$ gives

$$\eta^2 f'' + \eta f' + (\eta^2 - 1)f = 0, \qquad f[a\delta^{-1}(1+\mathrm{i})] = \Omega a.$$

We recognize here the equation for a modified Bessel function, which has two solutions I_1 and K_1. K_1 is not physical since it is infinite at $\eta = 0$. Therefore $f(\eta) = cI_1(\eta)$. Substituting for η and applying the boundary condition at the wall gives

$$\phi(r) = \Omega a \frac{I_1[r\delta^{-1}(1+\mathrm{i})]}{I_1[a\delta^{-1}(1+\mathrm{i})]}$$

and

$$u_\theta(r,t) = \Omega a \mathrm{Re}\left\{\frac{I_1[r\delta^{-1}(1+\mathrm{i})]}{I_1[a\delta^{-1}(1+\mathrm{i})]} e^{\mathrm{i}\omega t}\right\}.$$

The characteristic length δ gives the size of the boundary layer that forms along the cylinder wall. For large δ, (very viscous), we find solid body rotation. For small δ, there is a narrow boundary layer near the wall as in the two-dimensional problem treated in class. See Figure 1.

3 The equation for the velocity field is, assuming stationary flow with no body force,

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}, \qquad \mathbf{u}(x, y=0) = 0, \qquad \mathbf{u}(x, y\to\infty) \sim (\alpha x, -\alpha y).$$

Take the curl of the Navier Stokes equation to obtain the scalar vorticity equation

$$(\mathbf{u}\cdot\nabla)\omega = \nu\nabla^2\omega, \qquad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

It is enough to check that $\mathbf{u}$ is a solution of the vorticity equation since no constraint on the pressure is imposed.

Using the given form for $\mathbf{u}$, $\omega = -\alpha x\sqrt{\alpha/\nu}f''$. The vorticity equation becomes

$$\left[\alpha x f' \frac{\partial}{\partial x} - \alpha f \frac{\partial}{\partial \eta}\right]\left(-\alpha\sqrt{\frac{\alpha}{\nu}}f''\right) = \nu\frac{\alpha}{\nu}\left(-\alpha\sqrt{\frac{\alpha}{\nu}}x f'''\right)$$

which simplifies to

$$f^{(iv)} + ff''' - f'f'' = 0 \qquad (1)$$

At $\eta = 0$, $\mathbf{u} = 0$. Therefore, $f(0) = f'(0) = 0$. At infinity, $u \sim \alpha x$ imposes $f'(\infty) = 1$, which is consistent with $v \sim -\alpha y$ ($f \sim \eta$ at infinity). The governing equation (1) can be rewritten as

$$(f''' - f'^2 + ff'')' = 0, \qquad \text{or } f''' - f'^2 + ff'' = c$$

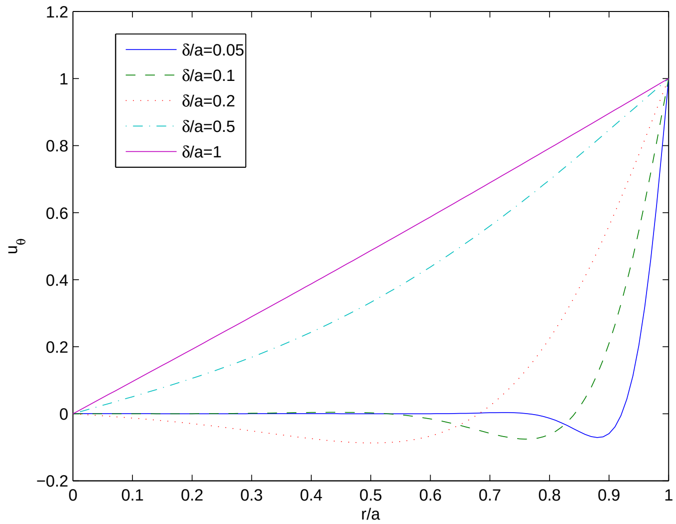


Figure 1: $u_\theta(r,t)/u_\theta(r,a)$ for different values of the characteristic distance $\delta = \sqrt{2\nu/\omega}$

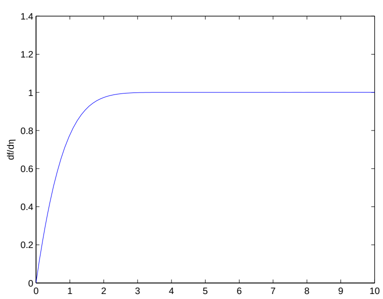
At infinity, $f \sim \eta$, therefore $f'''(\infty) = f''(\infty) = 0$ and $c = -1$. Finally, the given velocity field is a solution if and only if

$$f''' - f'^2 + ff'' + 1 = 0, \qquad f(0) = f'(0) = 0, \qquad f'(\infty) = 1$$

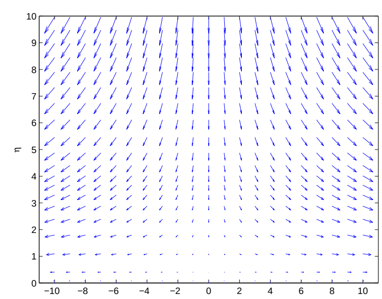
This is a two-points boundary value problem that can be solved using a shooting method: guess an initial value for $f''(0)$, integrate the initial value problem for f, check $f'(\infty)$ and iterate. The shooting is very sensitive to the initial guess... (try $f''(0) = 1.23$). See Figure 2(a).

The outside flow is the stagnation point flow we have seen previously. Viscosity leads to a boundary layer near the wall. The size of the boundary layer is of order $\sqrt{\nu/\alpha}$. We found $f'(3) = 0.998$, so at a distance of $3\sqrt{\nu/\alpha}$ from the wall, the horizontal velocity is 99.8% of its outer value. See velocity field in Figure 2(b)

4 The dimensional quantities for the problem are: characteristic velocity U (for example imposed by boundary condition), thickness h, dynamic viscosity μ, gravity g, surface tension σ and density ρ. There are three non dimensional parameters: Reynolds number $Re = \rho U h/\mu$, Froude number $Fr = U/\sqrt{gh}$ and Weber number $We = \rho U^2 h/\sigma$.



(a) Profile of $f'(\eta)$ for problem 3 obtained numerically with the shooting method



(b) Velocity field for problem 3