MAE210A: Fluid Mechanics I Fall Quarter 2006
http://maecourses.ucsd.edu/mae210a

Final Solutions

1 We calculate

$$\nabla\cdot\mathbf{u} = y + 1 + \frac{2z}{1+t^2}$$

and

$$\nabla\times\mathbf{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = (0, 0, -x).$$

The equations for the pathline of a particle releasde from $(1, 1, 1)$ at $t = 0$ are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xy,\ x(0) = 1, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y,\ y(0) = 1, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1+z^2}{1+t^2},\ z(0) = 1.$$

The second equation can be integrated directly: $y(t) = e^t$. Plugging this result in the first equation and solving for $x(t)$ leads to $x(t) = \exp\left(e^t - 1\right)$. The third equation must be solved independently. Using separation of variables, $\mathrm{d}t/(1 + t^2) = \mathrm{d}z/(1 + z^2)$. Integrating from $t = 0$ (where $z = 1$) to t gives $\arctan z = \arctan t + \pi/4$. Using trigonometric relations,

$$z = \tan\left(\arctan t + \frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{4} + \tan(\arctan t)}{1 - \tan\frac{\pi}{4}\tan(\arctan t)} = \frac{1+t}{1-t}.$$

The trajectory of this particle is therefore

$$\mathbf{x} = \left(\exp\left(e^t - 1\right), e^t, \frac{1+t}{1-t}\right).$$

If a particle is released from the origin, $u = v = 0$ and at all times $x = y = 0$ for this particle. Hence such a particle moves along the z-axis. On this axis, $\mathbf{u} = \big(0, 0, (1 + z^2)/(1 + t^2)\big)$ so $\mathbf{u}$ is parallel to the axis and the z-axis is a streamline. Therefore a particle released from the origin moves along a streamline.

2 By conservation of volume of incompressible liquid in the glass the total height doesn't change and remain in the stirred case $d_1 + d_2$. In both cases, (before and after stirring), we assume that the fluid is at rest. The pressure is therefore hydrostatic: $\mathrm{d}p/\mathrm{d}z = -\rho g$. Integrating from the bottom of the glass to the top, the pressure at the bottom of the glass is given by $p_b = p_0 + g\int_0^{d_1+d_2}\rho(z)\mathrm{d}z$.

Before stirring we have two layers of homogeneous density,

$$p_{b1} = p_0 + g(\rho_1 d_1 + \rho_2 d_2) = p_0 + g(d_1 + d_2)\left((1-\alpha)\rho_1 + \alpha\rho_2\right)$$

where $\alpha = d_2/(d_1 + d_2)$ is the fraction of the height occupied by the vermouth. The system is stably stratified so the lighter fluid (vodka) is on top: $\rho_2 > \rho_1$.

After stirring, there is one homogeneous layer of density ρ_e and $p_{b2} = p_0 + \rho_e g(d_1 + d_2)$. The conservation of mass gives $\rho_e V = \rho_1(V - V_2) + \rho_2 V_2$ with V the total volume and V_2 the volume of vermouth. V and V_2 are the volumes of 2 cones. The volume of a cone scales like the cube of its height therefore $V_2/V_1 = \alpha^3$, and $\rho_e = \rho_1(1 - \alpha^3) + \rho_2\alpha^3$.

$$\frac{p_{b1} - p_{b2}}{g(d_1 + d_2)} = (1 - \alpha)\rho_1 + \alpha\rho_2 - (1 - \alpha^3)\rho_1 - \alpha^3\rho_2 = \alpha(1 - \alpha^2)(\rho_2 - \rho_1) > 0$$

The pressure at the base of the glass goes down when the cocktail is stirred. Phisically, in the unstirred case, the heavier fluid is concentrated in the base. By stirring heavy fluid above the base is replaced by lighter fluid.

Two forces are applied to the grain: the buoyancy force $(\rho_e - \rho_3)gV_g$ downward and the Stokes drag $3\pi\mu lU$ where l is the diameter of the spheric grain ($V_g = \frac{\pi}{6}l^3$) and U the velocity. Newton's second law for the grain is:

$$\rho_3 V_g \dot{U} = (\rho_e - \rho_3)gV_g - 3\pi\mu lU.$$

The terminal velocity corresponds to the permanent regime when $\dot{U} = 0$ and the forces balance: $3\pi\mu lU = (\rho_e - \rho_3)\frac{\pi g l^3}{6}$. The terminal velocity is $U = g(\rho_3 - \rho_e)l^2/18\mu$.

3 In the local basis, $\mathbf{u}$ is parallel to $\mathbf{t}$ by definition: $\mathbf{u} = u\mathbf{t}$ and the operator $\mathbf{u}\cdot\nabla$ becomes $u\frac{\partial}{\partial s}$. The steady Euler equation with constant density is $(\mathbf{u}.\nabla)\mathbf{u} = -\nabla p/\rho - \nabla\Omega$ with the conservative body force $\mathbf{g} = -\nabla\Omega$. Hence

$$u\frac{\partial}{\partial s}(u\mathbf{t}) = -\frac{1}{\rho}\nabla p - \nabla\Omega.$$

The left-hand side can be expanded using $\partial\mathbf{t}/\partial s = \mathbf{n}/R$ where R is the local algebraic curvature radius of the streamline, giving

$$u\frac{\partial u}{\partial s}\mathbf{t} + \frac{u^2}{R}\mathbf{n} = \frac{\partial}{\partial s}\left(\frac{u^2}{2}\right)\mathbf{t} + \frac{u^2}{R}\mathbf{n} = -\frac{1}{\rho}\nabla p - \nabla\Omega.$$

Projecting onto $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ respectively leads to

$$\frac{\partial}{\partial s}\left(\frac{u^2}{2} + \frac{1}{\rho}\frac{p}{\rho} + \Omega\right) = 0, \qquad \frac{u^2}{R} = -\frac{\partial}{\partial n}\left(\frac{p}{\rho} + \Omega\right), \qquad \frac{\partial}{\partial b}\left(\frac{p}{\rho} + \Omega\right) = 0$$

The first equation is the steady Bernoulli equation (the Bernoulli function is constant on the streamline). The second equation shows that the centripetal acceleration is balanced by the normal gradient of pressure and potential. The last equation indicates that there is no net force applied to the fluid in the binormal direction.

4 The flow is radial and spherically symmetric, so $\mathbf{u} = u_r \mathbf{e}_r$. The vorticity of such a flow vanishes, so there is a velocity potential, $\phi(r)$, satisfying Laplace's equation. At the surface of the bubble the normal velocity of the fluid is equal to the velocity of the boundary, i.e. $\partial\phi/\partial r = \dot{R}$ at $r = R(t)$. Assuming no flow at infinity, $\partial\phi/\partial r \to 0$ as $r \to \infty$. Hence ϕ satisfies the problem

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 0, \qquad \frac{\partial \phi}{\partial r}(r = \infty) = 0, \qquad \frac{\partial \phi}{\partial r}(r = R(t)) = \dot{R}(t).$$

Solving the differential equation for ϕ gives $\phi = \alpha(t)/r + \beta(t)$. The function $\beta(t)$ can be set to zero (this does not change the velocity field). Applying the boundary condition at the surface of the bubble gives $\alpha(t) = -R^2\dot{R}$. This leads to $\phi(r,t) = -R^2\dot{R}/r$.

The irrotational Bernoulli theorem states that $p/\rho + \mathbf{u}^2/2 + \partial\phi/\partial t$ is a constant everywhere in the fluid. At infinity, $\partial\phi/\partial t$ and $\partial\phi/\partial r$ are zero and $p = p_\infty$, so

$$p = p_\infty - \rho\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial r}\right)^2\right] = p_\infty + \rho\left(\frac{2R\dot{R}^2 + R^2\ddot{R}}{r} - \frac{R^4\dot{R}^2}{2r^4}\right).$$

(i) If the pressure is negligible inside the bubble, the boundary condition at the surface of the bubble is p. Applying the previous relation at $r = R(t)$ leads to:

$$0 = \frac{p_\infty}{\rho} + R\ddot{R} + \frac{3}{2}\dot{R}^2$$

[Extra: we can rewrite this equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2p_\infty R^3}{3\rho} + R^3\dot{R}^2\right) = 0.$$

Integrating in time with initial radius R_0 and no initial velocity gives

$$\dot{R}^2 = \frac{2p_\infty}{3\rho}\left(\frac{R_0^3}{R^3} - 1\right)$$

Since the radius of the bubble is decreasing

$$\dot{R} = -\sqrt{\frac{2p_\infty}{3\rho}\left(\frac{R_0^3}{R^3} - 1\right)}.\Bigg]$$

(ii) Assuming adiabatic behavior for the gas, $p\rho^{-\gamma}$ is a constant. By conservation of mass in the bubble, $\rho/\rho_0 = (R_0/R)^3$ and $p = p_0(R_0/R)^{3\gamma}$. The equation of motion for $R(t)$ is then

$$p_\infty + \rho\left(\frac{3}{2}\dot{R}^2 + R\ddot{R}\right) = p_0\left(\frac{R_0}{R}\right)^{3\gamma}.$$

5 The velocity is assumed axial in both layers $\mathbf{u}_i = u_i \mathbf{e}_z$ with $i = 1, 2$ refers to the two different layers. The flow is fully developped, axisymmetric and steady (the velocity does not depend on θ, z or t). Therefore $\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = 0$, the Navier Stokes equations become:

$$\begin{cases} 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\frac{\mathrm{d}u}{\mathrm{d}r}) \\ 0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} \\ -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \end{cases}$$

Therefore the pressure is only a function of x. The first equation shows that the pressure gradient is only a function of r, therefore the pressure gradient is a constant $\mathrm{d}p/\mathrm{d}x = -\Delta p/L$ with Δp the pressure drop over a distance L. Let's define $P_x = -\Delta p/\rho L$. Integrating the axial component of Navier Stokes in each layer gives

$$u_i(r) = -\frac{P_x r^2}{4\mu_i} + A_i \log r + B_i.$$

The velocity also satisfies the following boundary conditions

$$u_1(0) = \text{finite}, \qquad u_1(a) = u_2(a), \qquad \mu_1 \frac{\mathrm{d}u_1}{\mathrm{d}r}(a) = \mu_2 \frac{\mathrm{d}u_2}{\mathrm{d}r}(a), \qquad u_2(b) = 0$$

The first condition imposes that $A_1 = 0$. The third condition becomes $-P_x a/2 = -P_x a/2 + \mu_2 A_2/a$ leading to $A_2 = 0$. The fourth condition then gives $B_2 = P_x b^2/4\mu_2$. Substitution in the second condition gives $A_1 = P_x a^2/4\mu_1 - P_x(a^2 - b^2)/4\mu_2$. Finally the velocity profile in the pipe is:

$$\begin{array}{ll} u_1(r) = \frac{P_x(a^2-r^2)}{4\mu_1} + \frac{P_x(b^2-a^2)}{4\mu_2}, & \text{for } 0 \le r \le a \\ u_2(r) = \frac{P_x(b^2-r^2)}{4\mu_2}, & \text{for } a \le r \le b \end{array}$$

The volume flux in the pipe is $Q = 2\pi \int_0^b u(r) r \mathrm{d}r$.

$$\begin{aligned} Q &= 2\pi \left[\int_0^a \left(\frac{P_x(a^2-r^2)}{4\mu_1} + \frac{P_x(b^2-a^2)}{4\mu_2} \right) r\mathrm{d}r + \int_a^b \frac{P_x(b^2-r^2)}{4\mu_2} r\mathrm{d}r \right] \\ &= 2\pi \left[\frac{P_x a^4}{16\mu_1} + \frac{P_x(b^2-a^2)a^2}{8\mu_2} + \frac{P_x}{4\mu_2}\left(\frac{b^2}{2}(b^2-a^2) - \frac{1}{4}(b^4-a^4) \right) \right] \\ &= 2\pi P_x \left[\frac{a^4}{16\mu_1} + \frac{b^4-a^4}{16\mu_2} \right] \end{aligned}$$

The shear stress is $\tau = \mu \mathrm{d}u/\mathrm{d}r$. Evaluated at the wall, $\tau_w = -P_x b/2$. This result could have been obtained writing the force balance on a cylindrical section of the fluid of thickness $\mathrm{d}x$. The pressure force $P_x \mathrm{d}x \pi b^2 \mathbf{e}_x$ is balanced by the viscous friction at the wall $2\pi b \mathrm{d}x \tau_w \mathbf{e}_x$. Note that the shear stress is only a function of the pressure gradient applied and not of the viscosity of the fluids.

6 We assume a relation of the form $t = f(l, d, D, \mu, \Delta\rho)$. There are 6 dimensional quantities and 3 independent units. From the π theorem, there are 3 dimensionless parameters, for example: $t\mu/d^2\Delta\rho$, d/D and d/l. The relation simplifies into $t = \frac{d^2\Delta\rho}{\mu} g(d/D, l/D)$. The parameters d, D and l depend only on the experimental setup. Assuming that we have done the measurement for a know fluid, we have been able to determine $g(d/D, l/D)$. Using a different fluid of know density, repeating the experiment we can compute μ.