Vector calculus

Use Cartesian coordinates $\mathbf{r} = \mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$; $r = |\mathbf{x}| = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. The vectors $\mathbf{a}$ and $\mathbf{b}$ have components (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively. In suffix notation, they are a_i and b_i. We take $f(\mathbf{x})$ to be a scalar function of $\mathbf{x}$ and $\mathbf{u}(\mathbf{x}) = (u_1, u_2, u_3)$ to be a vector function of $\mathbf{x}$; the latter is also $u_i(x_j)$.

Dot product: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i$. The dot product of two vectors is a *scalar*. Repeated indices are *summed over*.

Cross product: $\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$. The cross product of two vectors is a *vector*. In suffix notation, $(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$, where $\epsilon_{ijk} = 1$ if (i, j, k) are an even permutation of $(1, 2, 3)$, -1 if they are an odd permutation and 0 otherwise.

Gradient:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right); \qquad (\nabla f)_i = \frac{\partial f}{\partial x_i}$$

The gradient of a scalar function is a *vector*.

Divergence:

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_i}{\partial x_i}$$

The divergence of a vector function is a *scalar*.

Curl:

$$\nabla \times \mathbf{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right); \qquad (\nabla \times \mathbf{u})_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

The curl of a vector function is a *vector*.

Differentials:

$$\mathrm{d}f = \frac{\partial f}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\,\mathrm{d}x_2 + \frac{\partial f}{\partial x_3}\,\mathrm{d}x_3 = \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i$$

Divergence theorem:

$$\int_V \frac{\partial f}{\partial x_i}\,\mathrm{d}V = \int_S f n_i \,\mathrm{d}S$$

In this expression, $\mathbf{n}$ is the unit vector oriented outward from the volume V. The scalar function f can be replaced by a vector or a tensor.

Stokes' theorem:

$$\int_S \epsilon_{ijk} \frac{\partial f}{\partial x_j} n_i \,\mathrm{d}S = \int_C f \mathrm{d}l_k$$

Here C is any curve bounding the open surface S and $\mathbf{l}$ is the tangent vector to C. Again the scalar function f can be replaced by a vector or a tensor.

Tensors

Vectors are more than just three numbers. Their components also transform in a certain way, and the generalization of this law leads to tensors.

Transformation law: the components of a vector transform according to $a'_i = l_{ij}a_j$, where l_{ij} are components of the orthogonal matrix, i.e. $LL^T = I$, and the operation on the right is just matrix-vector multiplication.

Tensors: tensors generalize vectors. They are objects whose components transform according to $t_{ij\ldots k} = l_{ip}l_{jq}\ldots l_{kr}t_{pq\ldots r}$. The *order* of a tensor is the number of subscripts it has. A scalar is a zeroth-order tensor; a vector is a first-order tensor. A second-order tensor can be written down in components as a matrix. The *trace* of a second-order tensor is t_{ii}. *Contracting* a tensor, e.g. t_{ijijl} gives a tensor of lower order. The product of two tensors, e.g. $t_{ij}s_{kl}$ is another tensor.

Symmetry, antisymmetry and decomposition: a symmetric tensor satisfies $s_{ij} = s_{ji}$. An antisymmetric tensor satisfies $a_{ij} = -a_{ij}$. Any tensor can be decomposed into symmetric trace-less, antisymmetric and isotropic parts:

$$t_{ij} = [\tfrac{1}{2}(t_{ij} + t_{ji}) - \tfrac{1}{3}t_{ii}\delta_{ij}] + \tfrac{1}{2}(t_{ij} - t_{ji}) + \tfrac{1}{3}t_{ii}\delta_{ij}$$

(The $1/3$ comes from working in three dimensions.) An antisymmetric matrix can be written as $a_{ij} = \epsilon_{ijk}\omega_k$, where ω_k is a (pseudo-)vector.

Isotropic tensors: the only isotropic second-order tensor is δ_{ij}, which has the property that $\delta_{ij} = 1$ is $i = j$ and $\delta_{ij} = 0$ otherwise. Under transformation, we have $\delta'_{ij} = \delta_{ij}$. The only isotropic third-order (pseudo-)tensor is ϵ_{ijk} (the components of a pseudo-tensor do not change sign under reflection). The following identity is very important:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

Eigenvalues: A symmetric second-order tensor has orthogonal *principal axes*. These are the eigenvectors of the corresponding matrix.

Tensor calculus: the operator ∇ is a vector, hence ∇f is a vector, $\nabla\cdot\mathbf{u}$ is a scalar, and so on. Note the following important identity:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

Functions of r: The relation $r^2 = x_j x_j$ leads to

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}.$$

Orthogonal curvilinear coordinates: these are coordinate systems in which the basis vectors are mutually orthogonal, even though they may vary in space. E.g. cylindrical polars, spherical polars, as well as more exotic systems. See appendix of Batchelor.