Final Solutions

1 Use $\partial x_i/\partial x_j = \delta_{ij}$. Incompressibility:

$$\frac{\partial u_i}{\partial x_i} = a_{ii} = 0.$$

Irrotationality:

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}a_{kl}x_l = \epsilon_{ijk}a_{kj} = 0.$$

One can either multiply by ϵ_{pqi} or write out in suffices to find $a_{ij} = a_{ji}$. This means the matrix with elements a_{ij} is symmetric. In class, we saw that any flow can be decomposed locally into an antisymmetric part relate to vorticity and a symmetric part related to strain. If the flow is irrotational, the vorticity vanishes and the velocity gradient tensor is symmetric, i.e. $a_{ij} = a_{ji}$. The velocity potential is then $\phi = \frac{1}{2}a_{ij}x_ix_j$ as can be checked.

2 Particle path: solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (t+\cos t)y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 0,, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = -z.$$

The second equation shows that y is constant. Using the initial conditions one obtains

$$x(t) = \frac{t^2}{2} + \sin t + 1, \qquad y(t) = 1, \qquad z(t) = \mathrm{e}^{-t}.$$

Streaklines: solve the particle path equations with initial position $(1,1,1)$ at $t = t_*$, giving

$$x(t,t_*) = \frac{t^2}{2} + \sin t - \frac{t_*^2}{2} - \sin t_* + 1, \qquad y(t,t_*) = 1, \qquad z(t,t_*) = \mathrm{e}^{t_*-t}.$$

At $t = \pi$, these give

$$x(t_*) = \frac{\pi^2}{2} - \frac{t_*^2}{2} - \sin t_* + 1, \qquad y(t_*) = 1, \qquad z(t_*) = \mathrm{e}^{t_*-\pi}$$

for $0 \le t \le \pi$. Streamlines at $t = 0$: solve

$$\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}y}{0} = -\frac{\mathrm{d}z}{z} = \mathrm{d}s.$$

This gives $y = y_0$, $x = y_0 s + x_0$ and $z = \mathrm{e}^{-s}z_0$ with $s = 0$ corresponding to the point (x_0, y_0, z_0). This can also be written as

$$y = y_0, \qquad z = z_0\mathrm{e}^{-(x-x_0)/y_0}.$$

3 The upper surface is at z and has velocity v. The exit is at 0 and has velocity v_0. By mass conservation we have

$$\pi h(z)^2 v = \pi h(0)^2 v_0.$$

Apply Bernoulli's equation on a streamline from the upper surface to the outlet. Assume that at the outlet the flow comes out as a constant-diameter jet so that the pressure there is atmospheric. Then we have

$$gz + \tfrac{1}{2}v^2 = \tfrac{1}{2}v_0^2.$$

Eliminate v_0 from these two equations:

$$gz + \tfrac{1}{2}v^2 = \tfrac{1}{2}v^2 \frac{h(z)^4}{h(0)^4}.$$

We require v to be constant in time, so this is not an ordinary differential equation but just an algebraic relation. The profile $h(z)$ is given by

$$h(z) = h(0)\left(1 + \frac{2gz}{v^2}\right)^{1/4}.$$

4 For fully-developed flow, $\partial_x = \partial_\theta = 0$. The momentum equation reduces to

$$0 = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right),$$

where u is the velocity in the x-direction. The solution is $u = A + B\log r$. The boundary conditions give

$$u = U\frac{\log(r/b)}{\log(a/b)}.$$

The volume flux is

$$Q = \int_a^b U\frac{\log(r/b)}{\log(a/b)} 2\pi r\,\mathrm{d}r = U\pi\left(\frac{b^2 - a^2}{\log(a/b)} - a^2\right).$$

The average velocity is

$$\overline{U} = \frac{Q}{\pi(b^2 - a^2)} = U\left(\frac{1}{\log(a/b)} - \frac{a^2}{b^2 - a^2}\right).$$

The stress at the boundaries is $\mu \mathrm{d}u/\mathrm{d}r$, which is $\mu U r^{-1}(\log(a/b))^{-1}$. The force on the boundary is hence

$$-(2\pi bL)\frac{\mu U}{b\log(a/b)} - (2\pi aL)\frac{\mu U}{a\log(a/b)} = \frac{2\pi L\mu U}{\log(b/a)}.$$

It acts in the direction of U.

5 A is real so $\psi = 2Axy$. Streamlines are lines of constant $\psi = xy$, i.e. hyperbolae. The major and minor axes are the coordinate axes. The velocity field is $2A(x, -y)$, so if $A > 0$ the flow goes away from the y axis, and if $A < 0$ it goes away from the x axis. We have $|\boldsymbol{u}| = 2|A|\sqrt{x^2 + y^2} = 2|A|r$, so the speed is everywhere proportional to the distance from the origin. Call the sides 1, 2, 3 and 4 going anticlockwise and starting at the origin. The normal velocity is 0 for sides 1 and 4, so the mass and momentum fluxes through them are zero. For the other sides, the mass fluxes out are

$$Q_2 = \int_0^l \rho u \,\mathrm{d}y = \int_0^l \rho 2Al \,\mathrm{d}y = 2Al^2\rho, \qquad Q_3 = \int_0^l \rho v \,\mathrm{d}x = \int_0^l \rho(-2Al) \,\mathrm{d}y = -2Al^2\rho;$$

their sum is zero as expected. The momentum fluxes out in the x- and y-directions are

$$\begin{aligned} M_2 &= \int_0^l \rho u^2 \,\mathrm{d}y = \int_0^l \rho(2Al)^2 \,\mathrm{d}y = 4A^2l^3\rho, \\ M_3 &= \int_0^l \rho uv \,\mathrm{d}x = \int_0^l \rho(2Ax)(-2Al) \,\mathrm{d}y = -2A^2l^3\rho \end{aligned}$$

and

$$\begin{aligned} N_2 &= \int_0^l \rho uv \,\mathrm{d}y = \int_0^l \rho(2Al)(-2Ay) \,\mathrm{d}y = -2A^2l^3\rho, \\ M_3 &= \int_0^l \rho v^2 \,\mathrm{d}x = \int_0^l \rho(-2Al)^2 \,\mathrm{d}y = 4A^2l^3\rho. \end{aligned}$$

Their sum is not zero (so there must be other terms in the steady momentum balance).

6 Units: ω is inverse time $[T]^{-1}$ and λ is length $[L]$. The units of other quantities are

$$h : [L],\ g : [L][T]^{-2},\ T : [M][T]^{-2},\ \rho : [M][L]^{-3},\ \mu : [M][L]^{-1}[T]^{-1}.$$

We have seven parameters and three units (length, time and mass) so we can construct $7 - 3 = 4$ dimensionless groups. Four possible non-dimensional groups are

$$\omega\sqrt{\frac{h}{g}}, \qquad \frac{\lambda}{h}, \qquad \frac{T}{\rho g h^2}, \qquad \frac{\mu}{\rho\sqrt{gh^3}}.$$

Hence we have

$$\omega = \sqrt{\frac{g}{h}} F\left(\frac{\lambda}{h}, \frac{T}{\rho g h^2}, \frac{\mu}{\rho\sqrt{gh^3}}\right).$$