Solution V

1 If the flow is steady we have

$$\frac{1}{2}|\boldsymbol{u}|^2 + \int \frac{\mathrm{d}p}{\rho} = B,$$

along a streamline. If the flow is irrotational we have

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2 + \int \frac{\mathrm{d}p}{\rho} = B(t).$$

The interesting part is the pressure term. For an isentropic perfect gas we have $p = \kappa\rho^\gamma$, where κ and γ are constants. Then $\mathrm{d}p = \kappa\gamma\rho^{\gamma-1}\,\mathrm{d}\rho$ so

$$\int \frac{\mathrm{d}p}{\rho} = \kappa\gamma \int \rho^{\gamma-2}\,\mathrm{d}\rho = \frac{\kappa\gamma\rho^{\gamma-1}}{\gamma-1} = \frac{1}{\gamma-1}\frac{\gamma p}{\rho}.$$

The speed of sound squared is defined to be the derivative of pressure with respect to pressure (for some given thermodynamic state). For an isentropic gas,

$$c^2 \equiv \frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{\mathrm{d}(\kappa\rho^\gamma)}{\mathrm{d}\rho} = \gamma\kappa\rho^{\gamma-1} = \frac{\gamma p}{\rho}.$$

Hence

$$\int \frac{\mathrm{d}p}{\rho} = \frac{c^2}{\gamma-1}$$

and the result follows.

2 Assume the flow is quasi-static, i.e. the vessel is large enough for steady Bernoulli to apply at each instant. This means we also neglect the velocity of the free surface in the equation. Steady Bernoulli along a streamline connecting the free surface and the orifice then gives

$$gh = \tfrac{1}{2}v^2$$

where v is the velocity at the orifice. Conservation of volume then gives $vA = \dot{h}\pi(2Rh - h^2)$. Putting these together gives

$$\tfrac{1}{2}\dot{h}^2 A^{-2}\pi^2(2Rh - h^2)^2 = gh.$$

Separate variables:

$$\mathrm{d}t = -\frac{\pi}{A\sqrt{2g}}(2Rh^{1/2} - h^{3/2})\,\mathrm{d}h.$$

where we need a minus sign since h is decreasing as t increases. Integrate from 0 to t and h_1 to h_2 respectively. Then

$$t = \frac{\pi}{A\sqrt{2g}}\left[\frac{2}{3}R(h_1^{3/2} - h_2^{3/2}) - \frac{1}{5}(h_1^{5/2} - h_2^{5/2})\right].$$

3 Conservation of mass gives $v(z)A(z) = v_0A_0$, assuming the velocity is uniform across the stream. Steady Bernoulli along a streamline leads to $v(z)^2/2 + gz + p(z)/\rho = v_0^2 + p_0/\rho$. Take a streamline along the boundary, so $p(z) = p_0$ is atmospheric pressure. This gives

$$\frac{(v_0A_0)^2}{2A(z)^2} + gz = \frac{v_0^2}{2}.$$

This can be solved to give

$$A(z) = \frac{A_0}{\sqrt{(1 - 2gz/v_0^2)}}.$$

4 (i) The flow is inviscid and incompressible. It is radially symmetric so it is also irrotational. Hence the velocity potential ϕ exists and satisfies Laplace's equation

$$\nabla^2\phi = 0.$$

The spherically symmetric non-trivial solution to Laplace's equation is $\phi = Ar^{-1}$. The boundary condition at the surface of the bubble requires the normal velocity to be continuous. This gives $\dot{R} = -A/r^2$ at $r = R$, so

$$\phi = -\frac{R^2\dot{R}}{r}.$$

(ii) The flow is irrotational with constant density, and gravity does not act, so the quantity

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{p}{\rho} = B$$

is constant in the fluid. For large r, the velocity vanishes, and hence $B = p_\infty/\rho$, where p_∞ is the pressure far from the bubble. Computing the time-derivative of ϕ gives

$$p = p_\infty + \rho\left(\frac{R^2\ddot{R}}{r} + \frac{2R\dot{R}^2}{r} - \frac{R^4\dot{R}^2}{2r^4}\right).$$

(iii) If the pressure is neglected inside the bubble, $p = 0$ at the surface of the bubble $r = R$. Hence

$$0 = \frac{p_\infty}{\rho} + R\ddot{R} + \frac{3}{2}\dot{R}^2.$$

This equation can be integrated in time to give

$$C = \frac{p_\infty R^3}{3\rho} + \frac{1}{2} R^3 \dot{R}^2.$$

The constant C is fixed by taking $R = R_0$ when $\dot{R} = 0$. Separating variables and noting that $\dot{R}$ is negative gives

$$\mathrm{d}t = -\sqrt{\frac{3\rho}{2p_\infty}} \frac{\mathrm{d}R}{[(R_0/R)^3 - 1]^{1/2}}.$$

Integrating the left-hand side from 0 to t_c and the right-hand side from R_0 to 0 gives

$$t_c = \sqrt{\frac{3\rho}{2p_\infty}} \int_0^{R_0} [(R_0/R)^3 - 1]^{-1/2} \, \mathrm{d}R.$$