Solution VI

1 Write the complex potential in terms of real and imaginary parts:

$$w = \phi + \mathrm{i}\psi = (x + \mathrm{i}y)^2 + \frac{m}{2\pi} \log r\mathrm{e}^{\mathrm{i}\theta}$$

where $z = r\mathrm{e}^{\mathrm{i}\theta}$. This gives

$$\phi = x^2 - y^2 + \frac{m}{2\pi} \log r = r^2 \cos 2\theta + \frac{m}{2\pi} \log r, \qquad \psi = 2xy + \frac{m\theta}{2\pi} = r^2 \sin 2\theta + \frac{m\theta}{2\pi}.$$

Along any ray out from the origin, the second term in ψ is constant. The first term in ψ is 0 along the x- and y-axes. Hence ψ is constant along the axes and they are streamlines. The flux of fluid through the circle with radius a is

$$M = \int_0^{2\pi} \frac{\partial \phi}{\partial r} a \, \mathrm{d}\theta = \int_0^{2\pi} \left[2a \cos 2\theta + \frac{m}{2\pi a} \right] a \, \mathrm{d}\theta = m,$$

since the first term integrates to zero around the circle. The circulation is

$$\Gamma = \int_0^{2\pi} \frac{1}{r} \frac{\partial \phi}{\partial \theta} a \, \mathrm{d}\theta = \int_0^{2\pi} [-2a \sin 2\theta] a \, \mathrm{d}\theta = 0.$$

2 Write $z = c \cosh w$ and substitute in $w = \phi + \mathrm{i}\psi$. Then

$$x + \mathrm{i}y = c \left(\cosh \phi \cos \psi + \mathrm{i} \sinh \phi \sin \psi \right).$$

Equate real and imaginary parts and eliminate ϕ:

$$\frac{x^2}{c^2 \cos^2 \psi} - \frac{y^2}{c^2 \sin^2 \psi} = 1.$$

Taking ψ to be constant in this equation shows that streamlines are hyperbolae. The limits $\psi \to 0$ and $\psi \to \pi$ give the equations $x = \pm c \cos \psi$, so there are streamlines corresponding to the surface $y = 0$ for $|x| > 1$. Note also the the streamline $\psi = \pi/2$ is just $x = 0$. Figure 1 shows the streamlines. The complex velocity is given by

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{1}{\sqrt{z^2 - c^2}}.$$

At the edges of the plate $z = \pm c$, the velocity is unbounded.

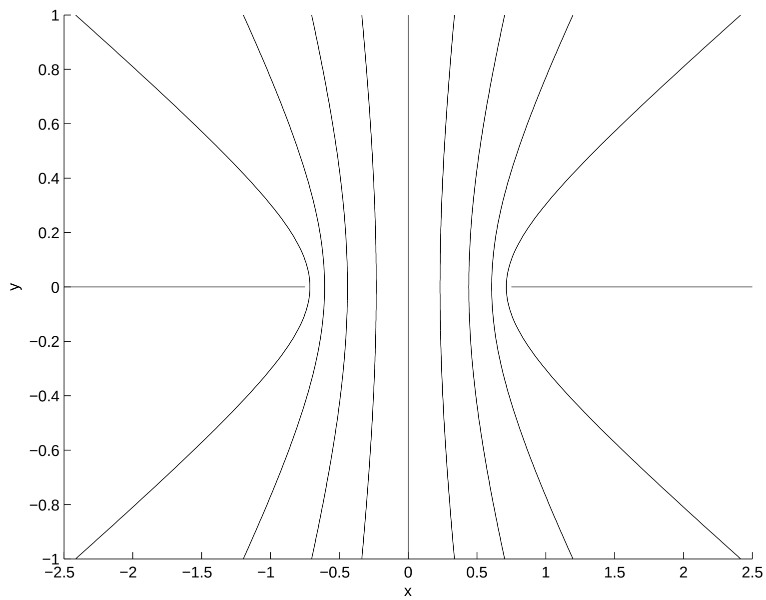

Figure 1: Streamlines for flow through aperture ($c = 0.75$).

3 This situation is like the usual case of flow past a cylinder, but now viewed from the lab frame, i.e. removing the constant flow at infinity. This suggests trying the potential $w = -Ua^2/z$ (the flow would be $-U$ at infinity). The complex velocity on the boundary is

$$u - \mathrm{i}v = \left.\frac{\mathrm{d}w}{\mathrm{d}z}\right|_{r=a} = U\mathrm{e}^{-2\mathrm{i}\theta}.$$

The velocity of the boundary is $(U, 0)$ and the normal vector is $\boldsymbol{n} = (\cos\theta, \sin\theta)$. Hence $\boldsymbol{U}\cdot\boldsymbol{n} = U\cos\theta$ and $\boldsymbol{u}\cdot\boldsymbol{n} = U(\cos 2\theta, \sin 2\theta)\cdot(\cos\theta, \sin\theta) = U\cos\theta$ so the normal velocity condition is satisfied. The kinetic energy of the fluid is

$$T = \tfrac{1}{2}\rho\int \boldsymbol{u}^2\,\mathrm{d}V = \tfrac{1}{2}\rho\int \frac{U^2a^4}{r^4}\,\mathrm{d}V = \tfrac{1}{2}\rho\int_a^\infty \frac{u^2a^4}{r^4}2\pi r\,\mathrm{d}r = \tfrac{1}{2}\rho\pi a^2U^2.$$

(This result can also be obtained from the surface integral derived in class. Careful of the direction of the normal vector.) The rate of change of kinetic energy of the cylinder plus the kinetic energy of the fluid is equal to the work done on the system, which is FU. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}MU^2 + \tfrac{1}{2}\rho\pi a^2U^2) = FU.$$

Define $M' = \rho\pi a^2$ and divide by U. Then

$$M\frac{\mathrm{d}U}{\mathrm{d}t} = F - M'\frac{\mathrm{d}U}{\mathrm{d}t},$$

The "added mass" M' is the mass of fluid (per unit length) displaced. One could also deduce this result by calculating the pressure on the cylinder using Bernoulli's equation and integrating over the cylinder to get the force.

4 Introduce an image vortex at $(0, -a)$ with anticlockwise circulation Γ. The complex potential is

$$w = Uz + \frac{i\Gamma}{2\pi}\log(z - ia) - \frac{i\Gamma}{2\pi}\log(z + ia).$$

(The sign is different from class because Kundu has positive Γ for clockwise circulation.) The complex velocity is then

$$\frac{dw}{dz} = U + \frac{i\Gamma}{2\pi(z - ia)} - \frac{i\Gamma}{2\pi(z + ia)}.$$

This has stagnation points on the x-axis at

$$U + \frac{i\Gamma}{2\pi(x - ia)} - \frac{i\Gamma}{2\pi(x + ia)} = U - \frac{\Gamma a}{\pi(x^2 + a^2)} = 0.$$

This gives $x = \pm a(\Gamma/\pi Ua - 1)^{1/2}$ so the are stagnation points only if $\Gamma > \pi Ua$. Figure 2 shows streamlines for three different cases. The third case shows that there is a dividing streamline and that the plate and vortices can be replaced by an ovoid body.

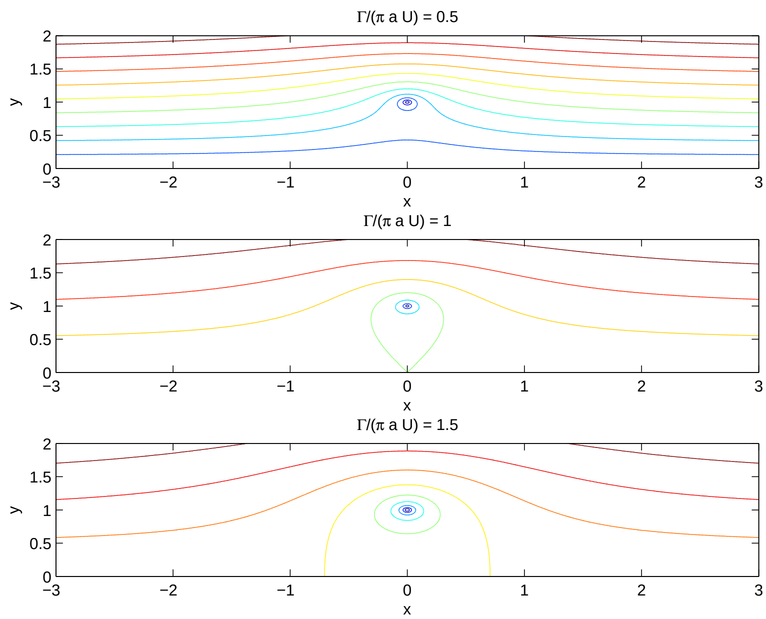

Figure 2: Streamlines for flow over plate with vortex.

Bernoulli along the plate gives

$$\frac{p}{\rho} + \tfrac{1}{2}\left(U - \frac{\Gamma a}{\pi(x^2 + a^2)}\right)^2 = \frac{p_\infty}{\rho} + \frac{1}{2}U^2.$$

This leads to

$$p_\infty - p = \frac{\rho\Gamma^2 a^2}{2\pi^2(x^2 + a^2)^2} - \frac{\rho U\Gamma a}{\pi(x^2 + a^2)}.$$

The upward force on the plate is

$$F = \int (p_\infty - p)\,\mathrm{d}x = \int_{-\infty}^{\infty} \left(\frac{\rho\Gamma^2 a^2}{2\pi^2(x^2+a^2)^2} - \frac{\rho U\Gamma a}{\pi(x^2+a^2)} \right)\,\mathrm{d}x = \frac{\rho\Gamma^2}{4\pi a} - \rho U\Gamma.$$

We have used

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+a^2} = \frac{\pi}{a}, \qquad \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)^2} = \frac{\pi}{2a^3}.$$