Solution VII

1 Since the tube is long, assume that the flow is fully developed. We apply a pressure difference between the two ends so the Navier–Stoke equation along the pipe takes the two forms

$$0 = -p_x + \frac{\mu_L}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_W}{\mathrm{d}r}\right), \qquad 0 = -p_x + \frac{\mu_A}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_A}{\mathrm{d}r}\right).$$

We can integrate these two equations to get

$$u_L = \frac{p_x}{4\mu_L}r^2 + A + B\log r, \qquad u_A = \frac{p_x}{4\mu_A}r^2 + C + D\log r.$$

The boundary conditions at the pipe walls are $u_L = 0$ at $r = a$. At the interface between air and liquid at $r = h$, the velocity, u, and the shear stress, $\mu\mathrm{d}u/\mathrm{d}r$, are continuous. (The thickness of the liquid layer is $a - h$.) In addition the logarithmic term must vanish along the centerline of the pipe so $D = 0$. The remaining three conditions are

$$\frac{p_x}{4\mu_L}a^2 + A + B\log a = 0, \quad \frac{p_x}{4\mu_L}h^2 + A + B\log h = \frac{p_x}{4\mu_A}h^2 + C, \quad \frac{p_x}{2}h + \frac{\mu_L B}{h} = \frac{p_x}{2}h.$$

Note that in the shear stress relation, the pressure terms cancel. Hence $B = 0$ and we obtain

$$u_L = \frac{p_x}{4\mu_L}(r^2 - a^2), \qquad u_A = \frac{p_x}{4\mu_A}(r^2 - h^2) + \frac{p_x}{4\mu_L}(h^2 - a^2).$$

The volume flux in the liquid is

$$Q_L = \int_h^a u_L 2\pi r\,\mathrm{d}r = \frac{\pi p_x}{4\mu_L}(a^2 - h^2)^2$$

while the volume flux in the air is

$$Q_A = \int_0^a u_A 2\pi r\,\mathrm{d}r = \frac{\pi p_x}{4\mu_A}h^2\left[h^2 + \frac{2\mu_A}{\mu_L}(a^2 - h^2)\right].$$

Their ratio is hence

$$\frac{Q_A}{Q_L} = \frac{\mu_L}{\mu_A}\frac{1 + 2(\mu_A/\mu_L)(a^2/h^2 - 1)}{(1 - a^2/h^2)^2}.$$

Note that this does not depend on the pressure gradient, just on the ratio of viscosities μ_L/μ_A and the geometric ratio h/a.

2 Take the x-axis along the slope and the z-axis perpendicular to it. The flow is steady ($\partial_t = 0$) and fully-developed ($\partial_x = 0$). Then the x-momentum equations in the two layers become

$$0 = \frac{\mathrm{d}^2 u_{T,B}}{\mathrm{d}z^2} - \frac{\rho g}{\mu_{T,B}} \sin\theta.$$

The boundary conditions are no slip at the bottom: $u_B = 0$ at $z = 0$, no stress at the top of the upper layer (neglecting shear stress in the air): $\mu_T \mathrm{d}u_T/\mathrm{d}z = 0$ at $z = h_B + h_T$, and continuity of velocity and stress at the interface: $u_B = u_T$ and $\mu_B \mathrm{d}u_B/\mathrm{d}z = \mu_T \mathrm{d}u_T/\mathrm{d}z$ at $z = h_B$. Integrate and solve for the four constants using the four boundary and interfacial conditions:

$$\begin{aligned} u_B &= \frac{\rho g}{2\mu_B} \sin\theta [z^2 - 2(h_T + h_B)z], \\ u_T &= \frac{\rho g}{2\mu_T} g \sin\theta [z^2 - 2(h_T + h_B)z + h_B(2h_T + h_B)(1 - \mu_T/\mu_B)]. \end{aligned}$$

The velocity in the bottom layer does not depend on μ_T because the shear stress at the top of the bottom layer is

$$\mu_B \mathrm{d}u_B/\mathrm{d}z = -\rho g \sin\theta h_T,$$

which is determined only by the weight of the fluid in the top layer.

3 As in class, look for a solution of the form

$$u_\theta = \frac{\Gamma}{2\pi r} f(\eta),$$

where $\eta = r^2/4\nu t$. We obtain the same equation as in class, $f'' + f' = 0$, but with different boundary conditions. Now at $t = 0$ and as $r \to \infty$, $f \to 0$ (no initial vortex and no flow at infinity), while at $r = 0$, $f = 1$ (vortex now in flow). Hence $f = e^{-\eta}$. This gives the solution.

4 Initial conditions: $u(y, 0) = 0$. Boundary conditions: $u(-h, t) = u(h, t) = 0$. We write the velocity as the steady-state solution that we expect to see for large time plus a Fourier cosine series that satisfies the boundary conditions:

$$u(y, t) = -\frac{P}{\mu}(y^2 - h^2) + \sum_{n=0}^{\infty} a_n(t) \cos{[n + \tfrac{1}{2})\pi y/h]}.$$

Substituting this into the governing equation and cancelling off the pressure term, multiplying by $\cos{[(m + \frac{1}{2})\pi y/h}$ and integrating over the channel gives

$$\frac{\mathrm{d}a_m}{\mathrm{d}t} = -\frac{\nu(m + \frac{1}{2})^2 \pi^2}{h^2} a_m.$$

The solution to this set of equations is

$$a_n(t) = a_m(0) \mathrm{e}^{-\mu[(n+\frac{1}{2})\pi/2h]^2 t}.$$

It now remains to find $a_m(0)$ from the equation

$$0 = -\frac{P}{\mu}(y^2 - h^2) + \sum_{n=1}^{\infty} a_n(0) \cos\left[(n + \tfrac{1}{2})\pi y/h.\right.$$

Once again multiply by $\cos\left[(m + \tfrac{1}{2})\pi y/h\right.$ and integrate to get

$$a_m(0) = \frac{P}{\mu h}\int_{-h}^{h} \cos\left[(m + \tfrac{1}{2})\pi y/h(y^2 - h^2)\, \mathrm{d}y = -\frac{32(-1)^n h^2}{(2m+1)^3\pi^3}\frac{P}{\mu}.\right.$$

Velocity profiles are shown in Figure 1.

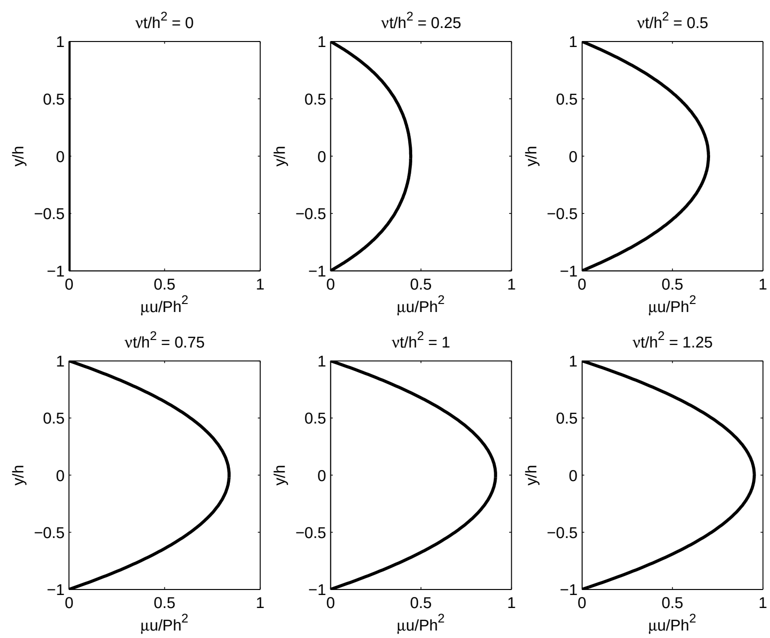

Figure 1: Temporal development of Poiseuille flow in channel.