Midterm

The MATLAB code is given separately.

1. Viscosity in the capillary instability of a thread See Figure 1. Note that the inviscid result uses a different scaling.

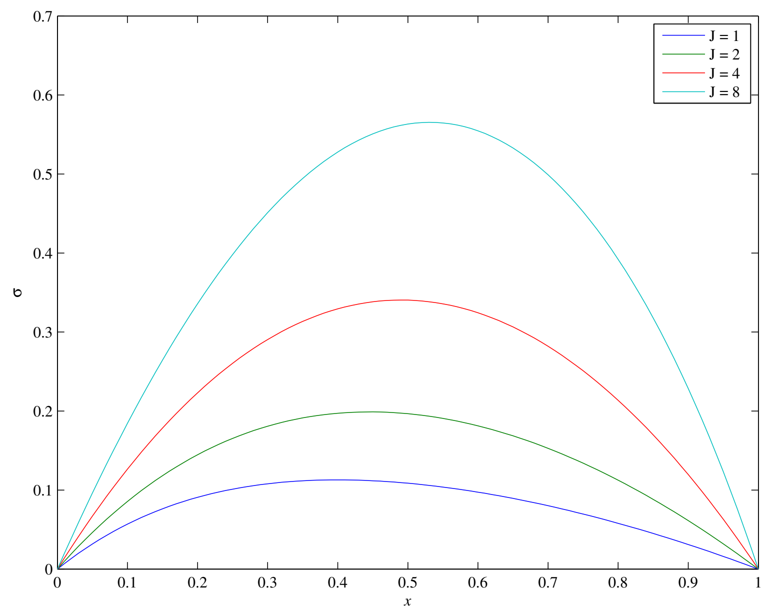

Figure 1: Nondimensional growth rate for $J = 1, 2, 4$ and 8.

2. The critical Rayleigh numbers for rigid-free and rigid-rigid boundary conditions The critical Reynolds numbers are $a = 3.116334360598749$, $R = 1707.761777134238$ for rigid-rigid and $a = 2.682326791289785$, $R = 1100.649606893142$ for rigid-free. The exact solution for free-free gives $a = 2.22144146907918$ and $R = 657.511364479516$. This is using Chandrasekhar's approach and I have given too many digits for fun. As expected, it is easier to drive convection with free boundaries, since the no-slip condition at rigid walls is more effective at inhibiting flow. Chandrasekhar's approach is fairly easy to program, although some care is required for the initial starting guess in the odd case. Note that Figure 2 in Chandrasekhar, which claims to show both the even and odd curves, must be wrong given the fact that the critical Rayleigh number for the odd case is around $17,600$: see Figure 2(b). Hence there is an unknown factor required to obtain the odd curve.

Figure 2(a) shows the critical curve for the three cases: rr, rf and ff. The curves have also been obtained by solving the underlying equations numerically (Chandrasekhar §—,12). In terms of W, the problem is

$$(\mathrm{D} - a^2)^3 W = -Ra^2 W$$

with boundary conditions $W = (\mathrm{D}^2 - a^2)^2 W = 0$ at $z = 0$ and 1 and either $\mathrm{D}W = 0$ or $\mathrm{D}^2 W = 0$ there (no slip and free slip respectively). Alternatively, in terms of Θ,

$$(\mathrm{D} - a^2)^3 \Theta = -Ra^2 \Theta$$

with boundary conditions $\Theta = (\mathrm{D}^2 - a^2)\Theta = 0$ at $z = 0$ and 1 and either $\mathrm{D}(\mathrm{D}^2 - a^2)\Theta = 0$ or $\mathrm{D}^2(\mathrm{D}^2 - a^2)\Theta = \mathrm{D}^4\Theta = 0$ there (no slip and free slip respectively). This formulation is more convenient in **3**.

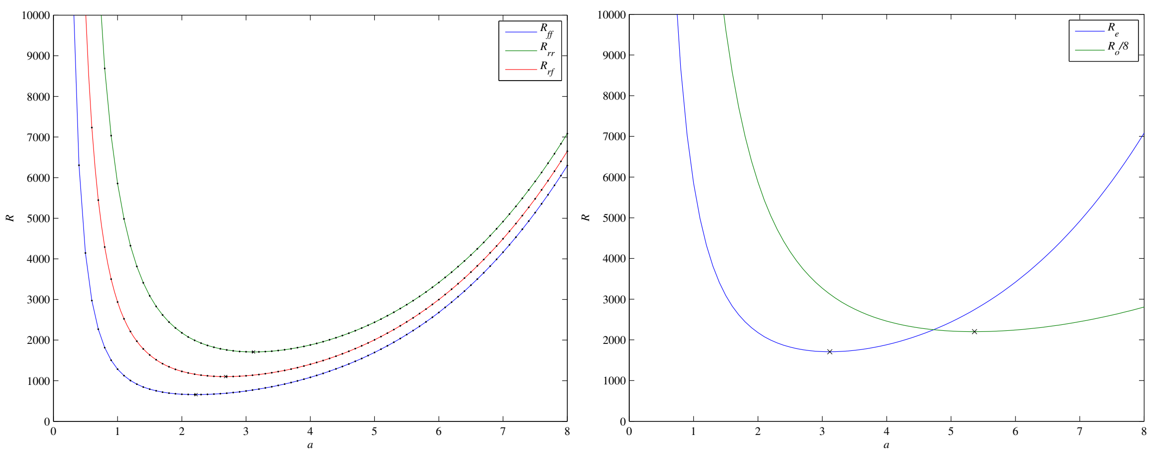

Figure 2: Left: critical curves computed by Chandrasekhar's method (solid curves), solving the W- and Θ-equations numerically (also black dots). The crosses correspond to the critical wavenumber and Rayleigh numbers. Right: an attempt to reproduce Figure 2 in Chandrasekhar.

3. A numerical Rayleigh–Bénard problem (i) The momentum equation becomes

$$\rho_0 \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \nabla p + \rho_0 \mathbf{g}[1 - \alpha(\rho - \rho_0)^3].$$

In the basic state, the temperature profile is linear so $\theta = \theta_0 + (\theta_1 - \theta_0)z/d$. Plugging this into the momentum equation and linearizing gives

$$\mathbf{u}'_t = -\frac{1}{\rho_0}\nabla p' - 3\alpha \mathbf{g} \frac{(\theta_1 - \theta_0)^2}{d^2} z^2 \theta' + \nu \nabla^2 \mathbf{u}';$$

the other equations are the same. Follow the usual process to get

$$(\partial_t - \nu\nabla^2)\nabla^2 w' = 3\alpha g \frac{(\theta_1 - \theta_0^2)}{d^2} z^2 \nabla_h^2 \theta', \qquad (\partial_t - \kappa\nabla^2)\theta' = \frac{(\theta_1 - \theta_0)}{h} w'.$$

Non-dimensionalizing and eliminating w' gives the equation in the text. At the boundary $\theta' = 0$. The boundary conditions $w' = 0$ along with $\theta' = 0$ and the linearized heat equation give $\Delta\theta' = 0$. Finally the no-stress condition gives $u'_z = v'_z = 0$, so continuity gives $w'_{zz} = 0$. Using the same argument as before leads to $\Delta^2\theta' = 0$.

(ii) Non-dimensionalize as in the text and substitute in the normal-mode solution $\theta' = \hat{T}(z)e^{\sigma t + i(kx+ly)}$ to get

$$[(\sigma + \kappa^2 - D^2)(\sigma Pr^{-1} + \kappa^2 - D^2)(D^2 - \kappa^2) + 3A\alpha^2 z^2]T = 0,$$

where $\kappa = (k^2 + l^2)^{1/2}$.

(iii) Multiply by T^* and integrate from 0 to d. Then the imaginary part shows that σ_i multiplied by a positive integral vanishes. Hence the principle of exchange of stabilities remains valid.

(iv) To solve the eigenvalue problem numerically, use the Θ formulation as above with $\Theta = T$, so that

$$(\mathrm{D} - a^2)^3\Theta = -3Az^2a^2\Theta$$

and the same boundary conditions as above. Figure 3 shows the corresponding critical curves. Roughly, the critical wavenumbers are $a = 3.2$, $R = 2012.9$ for rigid-rigid, $a = 2.7$ and $a = 1120.9$ for rigid-free, and $a = 2.2$ and $R = 763.7$ for free free.

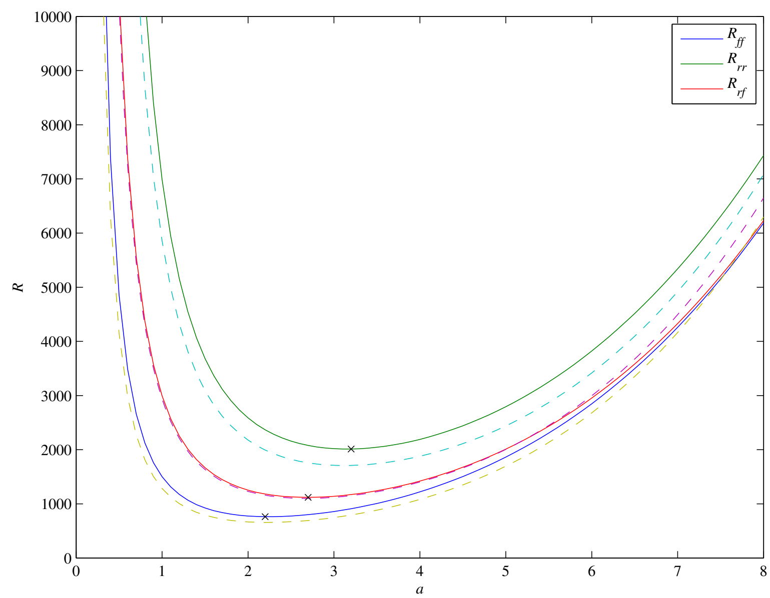

Figure 3: Left: critical curves for Hupnol computed from the Θ-equation numerically. The crosses correspond to the critical wavenumber and Rayleigh numbers.

4. Compressible Rayleigh–Bénard convection There was no correct answer for this. I was expecting to see a discussion of the governing equations, the relevant boundary conditions, the nondimensional parameters and an account of the form of the critical curves in terms of the appropriate Rayleigh number and wavenumber and their dependence on the other nondimensional parameters.