Solution I

Phase plane System: $\dot{x} = x(2-y-x)$, $\dot{y} = (x-2)y$. There are fixed points at $(0,0)$ and $(2,0)$. The Jacobian matrix is

$$\begin{pmatrix} 2-y-2x & -x \\ y & x-2 \end{pmatrix}.$$

The fixed point at $(0,0)$ has eigenvalues -2 and 2 with eigenvectors $(1\,0)^T$ and $(0\,1)^T$, so it is a saddle. The fixed point at $(2,0)$ has eigenvalues -2 and 0 with eigenvectors $(1\,0)^T$ and $(1\;-1)^T$, so it is a degenerate fixed point. In the the linearized flow, the flow is horizontal into the line $x+y=2$. Further analysis of the flow near this line requires the concept of a center manifold.

D 2.4 The fixed points are 0, 1 and R^{-1}. Differentiate to get $f'(u) = -1 + 2(1+R)u - 3Ru^2$. Hence $f'(0) = -1$ and $u = 0$ is a stable solution. For $u = 1$, find $f'(1) = 1 - R$, so the solution is stable for $R > 1$. For $u = R^{-1}$, find $f'(R^{-1}) = 1 - R^{-1}$ and the solution is stable for $R < 1$. If $u(0) = A$, the solution tends to 0 if $A < \min(1, R^{-1})$. So for large R, there is a smaller and smaller range of positive A which lead to decay.

D 2.18 Substitute $(x\,y)^T = (a\,b)^T \mathrm{e}^{-\mathrm{i}\omega t}$ into the governing equations. This gives the matrix relation

$$\begin{pmatrix} \mathrm{i}(\omega-\omega_1) & \epsilon p_1 \\ \epsilon p_2 & \mathrm{i}(\omega-\omega_2) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For this linear homogeneous solution to have solutions, the determinant of the matrix must vanish. This gives

$$\omega^2 - \omega(\omega_1+\omega_2) + \omega_1\omega_2 + \epsilon^2 p_1 p_2 = 0.$$

Solving the quadratic and substituting in for ω_2 gives

$$\omega = \omega_1 + \tfrac{1}{2}b\epsilon \pm \tfrac{1}{2}\epsilon(b^2 - 4p_1p_2)^{1/2}.$$

If $\epsilon = 0$, the quadratic has the double root $\omega = \omega_1$. One can check that this corresponds to two different linearly independent eigenvalues $(1\,0)^T$ and $(0\,1)^T$, and the solution is $(x\,y) = (x_0\,y_0)\,\mathrm{e}^{-\mathrm{i}\omega_1 t}$, i.e. neutrally stable with no algebraic instability, since there are two linearly independent eigenvectors. If $\epsilon > 0$ and $4p_1p_2 > b^2$, the contents of the square root are negative so the solutions for ω have non-zero imaginary part. In particular one root has positive imaginary part, which means that $\mathrm{e}^{-\mathrm{i}\omega t}$ increases with time and the null solution is unstable.

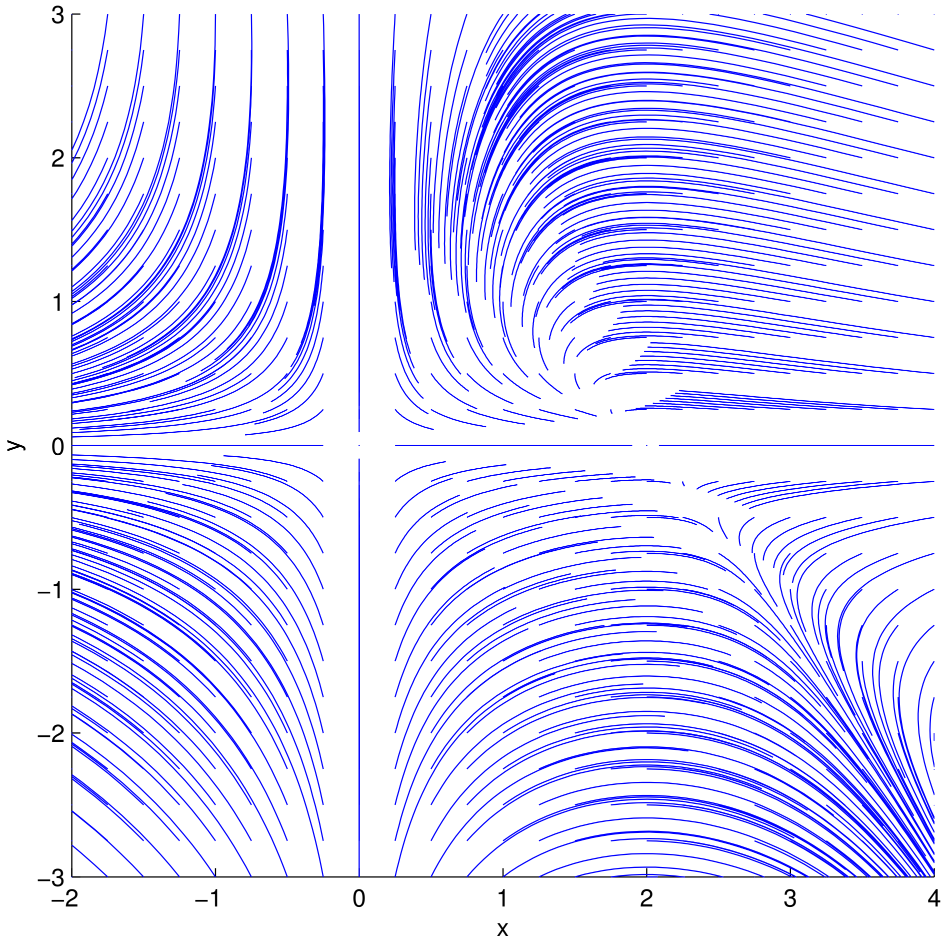

Figure 1: Phase plane (trajectories are integrated until $t = 0.5$).

C 1.6.7 The eigenvalues are $-\epsilon$ and -2ϵ with eigenvectors $\mathbf{e}_1 = (1\,0)^T$ and $\mathbf{u}_2 = (1, -\epsilon)^T$ respectively. The solution takes the form

$$\mathbf{x}(t) = y_{01}\mathrm{e}^{-\epsilon t}\mathbf{e}_1 + y_{20}\mathrm{e}^{-\epsilon t}\mathbf{e}_2.$$

Applying the initial condition gives the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 0 & -\epsilon \end{pmatrix} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}.$$

This can be solved to give

$$\mathbf{x}(t) = (x_{10} + x_{20}/\epsilon)\mathrm{e}^{-\epsilon t}\mathbf{e}_1 + (-x_{20}/\epsilon)\mathrm{e}^{-\epsilon t}\mathbf{e}_2.$$

For large times, the exponential terms both decay and $|\mathbf{x}(t)| \to 0$.
The energy is given by

$$E(t) = y_{10}^2\mathrm{e}^{-2\epsilon t} + 2y_{10}y_{20}\mathrm{e}^{-3\epsilon t} + y_{20}^2(1+\epsilon)^2\mathrm{e}^{-4\epsilon t}.$$

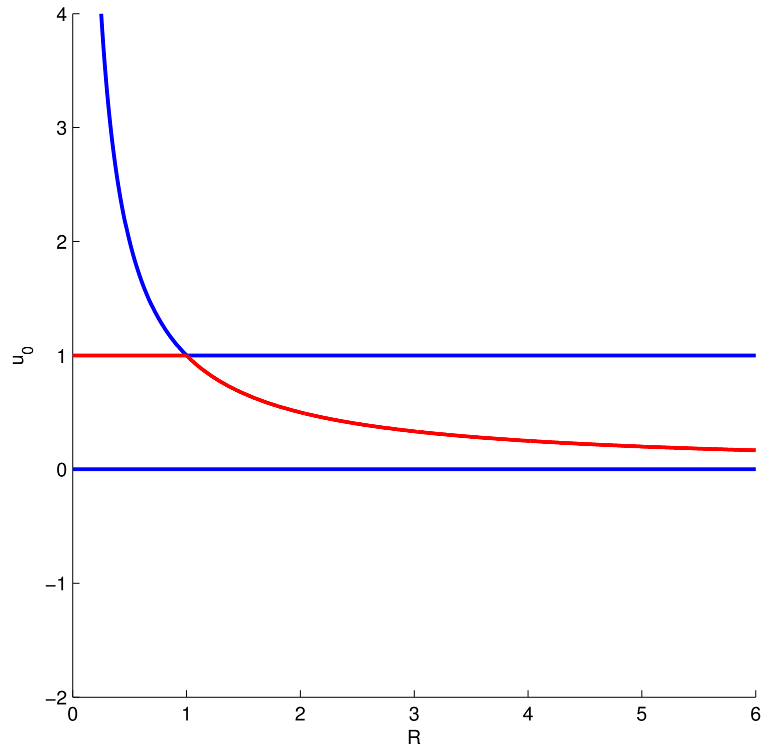

Figure 2: Bifurcation diagram. Blue portions of the curves are stable, red portions are unstable.

A Taylor expansion gives for small times

$$E(t) = (y_{10} + y_{20})^2 + (\epsilon y_{20})^2 - [2y_{10}^2 + 4(1+\epsilon)^2 y_{20}^2 + 6y_{10}y_{20}]\epsilon t + O(\epsilon^2 t^2).$$

To see when this can decay, write $y_{20} = -ay_{10}$. The ϵt term changes sign when the following quadratic vanishes:

$$2(1+\epsilon^2)a^2 - 3a + 1 = 0,$$

which has roots

$$a_{\pm} = \frac{3 \pm \sqrt{1-8\epsilon^2}}{4(1+\epsilon^2)}$$

Energy initially grows if the quadratic is negative, which is the case if a is not large in magnitude. Hence there is growth if $a_- < a < a_+$.

Differentiate to find time at which $E(t)$ passes through a maximum by solving the equation

$$-2y_{10}^2 e^{-2\epsilon t} - 6y_{10}y_{20}e^{-3\epsilon t} - 4y_{20}^2(1+\epsilon)^2 e^{-4\epsilon t} = 0.$$

Substituting in $y_{20} = -ay_{10}$ again gives

$$1 - 3ae^{-\epsilon t} + 2a^2(1+\epsilon)^2 e^{-2\epsilon t} = 0.$$

This is the same quadratic as before for $a\mathrm{e}^{-\epsilon t}$, so $a\mathrm{e}^{-\epsilon t} = a_\pm$. We need take a_- on the right-hand side, since $t > 0$. Hence

$$\epsilon t_{max} = \ln \frac{a}{a_-}.$$

Now plug into the formulas for E:

$$\frac{E(t_{max})}{E_0} = \frac{\mathrm{e}^{-2\epsilon t_{max}} - 2a\mathrm{e}^{-3\epsilon t_{max}} + a^2(1+\epsilon)^2\mathrm{e}^{-4\epsilon t_{max}}}{1 - 2a + a^2(1+\epsilon)^2} = \frac{a_-^2}{a^2}\frac{1 - 2a_- + a_-^2(1+\epsilon)^2}{1 - 2a + a^2(1+\epsilon)^2}.$$

In this fraction, the numerator is fixed. To maximize the ratio as a function of a, can minimize the denominator. The derivative of the denominator is

$$2a[1 - 3a + 2a^2(1+\epsilon)^2].$$

So the roots are 0 and $a_\pm$. The maximum comes from choosing a_+. To obtain the final expression, multiply the first fraction top and bottom by $2(1+\epsilon)^2$ and the second by a, then get rid of the $(1+\epsilon)^2$ terms by using the quadratic satisfied by a_pm:

$$\frac{E(t_{max})}{E_0} = \frac{3a_- - 1}{3a_+ - 1}\frac{1 - a_-}{1 - a_+}.$$