Solution I

Phase plane System: $\dot{x} = x(2 - y - x)$, $\dot{y} = (x - 2)y$. There are fixed points at $(0, 0)$ and $(2, 0)$. The Jacobian matrix is

$$
\begin{pmatrix}
2 - y - 2x & -x \\
y & x - 2
\end{pmatrix}.
$$

The fixed point at $(0, 0)$ has eigenvalues $-2$ and $2$ with eigenvectors $(1, 0)^T$ and $(0, 1)^T$, so it is a saddle. The fixed point at $(2, 0)$ has eigenvalues $-2$ and $0$ with eigenvectors $(1, 0)^T$ and $(1, -1)^T$, so it is a degenerate fixed point. In the the linearized flow, the flow is horizontal into the line $x + y = 2$. Further analysis of the flow near this line requires the concept of a center manifold.

D 2.4 The fixed points are $0, 1$ and $R^{-1}$. Differentiate to get $f'(u) = -1 + 2(1 + R)u - 3Ru^2$. Hence $f'(0) = -1$ and $u = 0$ is a stable solution. For $u = 1$, find $f'(1) = 1 - R$, so the solution is stable for $R > 1$. For $u = R^{-1}$, find $f'(R^{-1}) = 1 - R^{-1}$ and the solution is stable for $R < 1$. If $u(0) = A$, the solution tends to 0 if $A < \min(1, R^{-1})$. So for large $R$, there is a smaller and smaller range of positive $A$ which lead to decay.

D 2.18 Substitute $(x, y)^T = (a, b)^T e^{-i\omega t}$ into the governing equations. This gives the matrix relation

$$
\begin{pmatrix}
i(\omega - \omega_1) & ep_1 \\
e p_2 & i(\omega - \omega_2)
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \begin{pmatrix}0 \\ 0\end{pmatrix}.
$$

For this linear homogeneous solution to have solutions, the determinant of the matrix must vanish. This gives

$$
\omega^2 - \omega(\omega_1 + \omega_2) + \omega_1\omega_2 + \epsilon^2 p_1p_2 = 0.
$$

Solving the quadratic and substituting in for $\omega_2$ gives

$$
\omega = \omega_1 + \frac{1}{2}b\epsilon \pm \frac{1}{2}\epsilon(b^2 - 4p_1p_2)^{1/2}.
$$

If $\epsilon = 0$, the quadratic has the double root $\omega = \omega_1$. One can check that this corresponds to two different linearly independent eigenvalues $(1, 0)^T$ and $(0, 1)^T$, and the solution is $(x, y) = (x_0, y_0) e^{-i\omega_1 t}$, i.e. neutrally stable with no algebraic instability, since there are two linearly independent eigenvectors. If $\epsilon > 0$ and $4p_1p_2 > b^2$, the contents of the square root are negative so the solutions for $\omega$ have non-zero imaginary part. In particular one root has positive imaginary part, which means that $e^{-i\omega t}$ increases with time and the null solution is unstable.
The eigenvalues are \(-\epsilon\) and \(-2\epsilon\) with eigenvectors \(e_1 = (1, 0)^T\) and \(e_2 = (1, -\epsilon)^T\) respectively. The solution takes the form
\[
x(t) = y_{10}e^{-\epsilon t}e_1 + y_{20}e^{-2\epsilon t}e_2.
\]
Applying the initial condition gives the matrix equation
\[
\begin{pmatrix}
1 & 1 \\
0 & -\epsilon
\end{pmatrix}
\begin{pmatrix}
y_{10} \\
y_{20}
\end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}.
\]
This can be solved to give
\[
x(t) = (x_{10} + x_{20}/\epsilon)e^{-\epsilon t}e_1 + (-x_{20}/\epsilon)e^{-2\epsilon t}e_2.
\]
For large times, the exponential terms both decay and \(|x(t)| \to 0\).
The energy is given by
\[
E(t) = y_{10}^2e^{-2\epsilon t} + 2y_{10}y_{20}e^{-3\epsilon t} + y_{20}^2(1 + \epsilon)^2e^{-4\epsilon t}.
\]
Figure 2: Bifurcation diagram. Blue portions of the curves are stable, red portions are unstable.

A Taylor expansion gives for small times

\[ E(t) = (y_{10} + y_{20})^2 + (\epsilon y_{20})^2 - [2y_{10}^2 + 4(1 + \epsilon)^2 y_{20}^2 + 6y_{10}y_{20}] \epsilon t + O(\epsilon^2 t^2). \]

To see when this can decay, write \( y_{20} = -a y_{10}. \) The \( \epsilon t \) term changes sign when the following quadratic vanishes:

\[ 2(1 + \epsilon^2)a^2 - 3a + 1 = 0, \]

which has roots

\[ a_{\pm} = \frac{3 \pm \sqrt{1 - 8\epsilon^2}}{4(1 + \epsilon^2)} \]

Energy initially grows if the quadratic is negative, which is the case if \( a \) is not large in magnitude. Hence there is growth if \( a_- < a < a_+. \)

Differentiate to find time at which \( E(t) \) passes through a maximum by solving the equation

\[ -2y_{10}^2e^{-2\epsilon t} - 6y_{10}y_{20}e^{-3\epsilon t} - 4y_{20}^2(1 + \epsilon)^2e^{-4\epsilon t} = 0. \]

Substituting in \( y_{20} = -a y_{10} \) again gives

\[ 1 - 3ae^{-\epsilon t} + 2a^2(1 + \epsilon)^2e^{-2\epsilon t} = 0. \]
This is the same quadratic as before for \( ae^{-et} \), so \( ae^{-et} = a_\pm \). We need take \( a_- \) on the right-hand side, since \( t > 0 \). Hence

\[
\epsilon t_{\text{max}} = \ln \frac{a}{a_-}.
\]

Now plug into the formulas for \( E \):

\[
\frac{E(t_{\text{max}})}{E_0} = \frac{e^{-2\epsilon t_{\text{max}}} - 2ae^{-3\epsilon t_{\text{max}}} + a^2(1 + \epsilon)^2 e^{-4\epsilon t_{\text{max}}}}{1 - 2a + a^2(1 + \epsilon)^2} = \frac{a^2 1 - 2a_- + a_-^2 (1 + \epsilon)^2}{a^2 1 - 2a + a^2(1 + \epsilon)^2}.
\]

In this fraction, the numerator is fixed. To maximize the ratio as a function of \( a \), can minimize the denominator. The derivative of the denominator is

\[
2a[1 - 3a + 2a^2(1 + \epsilon)^2].
\]

So the roots are 0 and \( a_\pm \). The maximum comes from choosing \( a_+ \). To obtain the final expression, multiply the first fraction top and bottom by \( 2(1 + \epsilon)^2 \) and the second by \( a \), then get rid of the \( (1 + \epsilon)^2 \) terms by using the quadratic satisfied by \( a_p m \):

\[
\frac{E(t_{\text{max}})}{E_0} = \frac{3a_- - 11 - a_-}{3a_+ - 11 - a_+}.
\]