Solution II

D 4.1 The equations are exactly the same as before, with the difference that the pressure in the fluid is now p_∞ and the dynamic boundary condition becomes

$$p' = \gamma\left(\frac{\eta'}{a^2} + \frac{1}{a^2}\eta'_{\theta\theta} + \eta'_{zz}\right),$$

since the normal (which goes out of the fluid) has changed direction. The solution to Laplace's equation bounded at infinity is $\hat{p} = AK_n(\alpha r)$ (it should really be $|\alpha|$ so use $\alpha > 0$ to avoid that problem) and the only difference in the boundary conditions is the minus sign. Hence

$$s^2 = -\frac{\gamma}{a^3\rho}\frac{\alpha K'_n(\alpha)}{K_n(\alpha)}(1 - n^2 - \alpha^2).$$

Now $\alpha K'_n(\alpha)/K_n(\alpha) < 0$ for all α so modes are unstable if and only if $n = 0$ and $0 < \alpha < 1$. The maximum dimensionless growth rate can be found (e.g. using MATLAB) as $s(\rho a^3/\gamma)^{1/2} = 0.8201$ at $\alpha = 0.4839$ (see Figure 1).

```
alpha=0.01:0.01:1;
% note that f is minus s^2 to use matlab's fminbnd command; plot -f
f=inline('-sqrt(-alpha.*(-besselk(1,alpha))./besselk(0,alpha).*(1-alpha.^2))');
plot(alpha,-f(alpha))
xlabel('\alpha')
a=fminbnd(f,0,1)
-f(a)
```

D 6.10 (i) Substituting into the continuity equation gives

$$\nabla\cdot\mathbf{u} = 2^{1/2}(k^2+1)k^{-1}XkC_xC_z - 2^{1/2}(k^2+1)XC_xC_z = 0.$$

(ii) $S_z(0) = S_z(\pi) = S_{2z}(0) = S_{2z}(\pi) = 0$ so the boundary conditions are satisfied.
(iii) The nonlinear terms are

$$uu_x + wu_z = k^{-1}(k^2+1)^2X^2\sin 2kx, \qquad uw_x + ww_z = (k^2+1)^2X^2S_{2z}.$$

The vorticity is $q = -2^{1/2}(k^2+1)^2k^{-1}XS_xS_z$. The curl of the momentum equation takes the form

$$\frac{\partial q}{\partial t} + (\mathbf{u}\cdot\nabla)q = -\sigma\frac{\partial\theta}{\partial x} + \sigma\Delta q.$$

The nonlinear term $(\mathbf{u}\cdot\nabla)q$ is in fact zero. We read off

$$-2^{1/2}(k^2+1)^2k^{-1}X_t = \sigma(k^2+1)^3k^{-2}2^{1/2}Y(-k) - 2^{1/2}(k^2+1)^2k^{-1}X(-k^2-1).$$

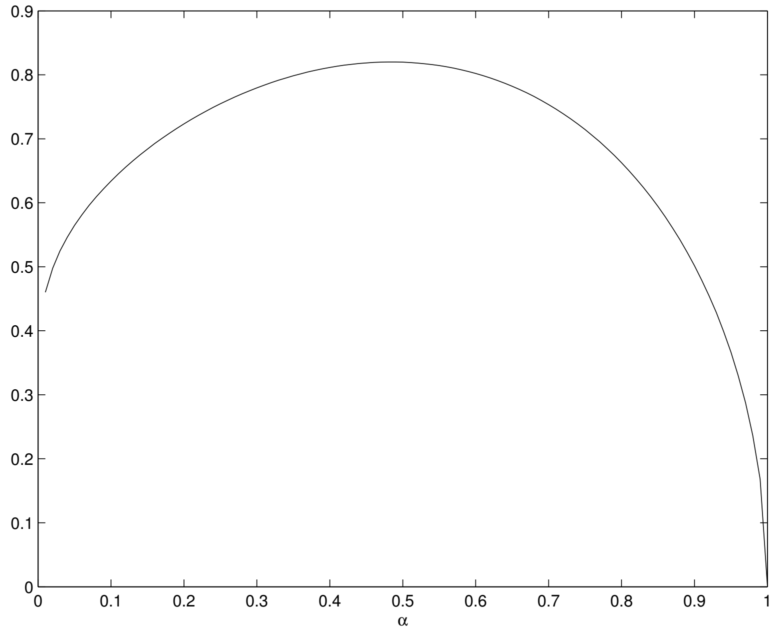

Figure 1: Nondimensional growth rate $s(\rho a^3/\gamma)^{1/2}$ for $n = 0$ and $0 < \alpha < 1$.

Simplifying gives

$$-\frac{\mathrm{d}X}{\mathrm{d}\tau} = \sigma(-Y + X).$$

(iv) The convective terms are

$$\begin{aligned}\mathbf{u}\cdot\nabla\theta &= (k^2+1)^4 k^{-2} X S_x C_z[-Y(-kC_x)S_z + YC_xC_z + 2^{1/2}Z(2C_{2z})] \\ &= (k^2+1)^4 k^{-2}(XYS_{2z} + 2^{3/2}ZXC_xS_zC_{2z}).\end{aligned}$$

Note that $S_zC_{2z} = \frac{1}{2}(S_{3z} - S_z)$, the first term of which will not appear in the truncations used here. Identifying the C_xS_z and S_{2z} terms in turn gives

$$\begin{aligned}& -(k^2+1)^3 k^{-2} 2^{1/2} Y_t + (k^2+1)^4 k^{-2} 2^{3/2} ZX(-\tfrac{1}{2}) \\ = \quad & -R2^{1/2}(k^2+1)X - (k^2+1)^3 k^{-2} 2^{1/2} Y(-k^2-1)\end{aligned}$$

and

$$-(k^2+1)^3 k^{-2} Z_t + (k^2+1)^4 k^{-2} XY = -(k^2+1)^3 k^{-2} Z(-4).$$

Simplifying gives

$$\begin{aligned}-\frac{\mathrm{d}Y}{\mathrm{d}\tau} - ZX &= -\frac{Rk^2}{(k^2+1)^3}X + Y, \\ -\frac{\mathrm{d}Z}{\mathrm{d}\tau} + XY &= \frac{4}{(k^2+1)}Z.\end{aligned}$$

C 2.8.4 1. Since K and μ_j are constants, $\mathbf{U}_j$ can be written as the gradient of a velocity potential Φ_j. The continuity equation then gives $\nabla^2\Phi_j = 0$.
2. Uniform ascension means $\mathbf{U}_j = (0, V)$. This corresponds to $\overline{\Phi}_j = Vy + C$, where C is a function of time. For convenience, use the new variable $z = y - Vt$ so that the unperturbed boundary between the fluids is at $z = 0$. Then $\overline{\Phi}_j = Vz$ and equating potential and pressure from Darcy's law gives the second relation.
3. From Darcy's law and continuity, the perturbation potential and pressure are both harmonic functions with $\nabla^2\phi_j = \nabla^2 p_j = 0$ (no linearization required). The conditions at infinity are decay for ϕ_j and p_j. The conditions at the front become (kinematic and Darcy's law)

$$\frac{\partial\phi_1}{\partial z} = \frac{\partial\phi_2}{\partial z} = \frac{\mathrm{D}h}{\mathrm{D}t}, \quad -\left(\frac{\mu_1 V}{K} + \rho_1 g\right) h + p_1 = -\left(\frac{\mu_2 V}{K} + \rho_2 g\right) h + p_2 \qquad \text{at } z = h(x,t).$$

Linearizing about $z = 0$ leads to

$$\frac{\partial\phi_1}{\partial z} = \frac{\partial\phi_2}{\partial z} = \frac{\partial h}{\partial t}, \quad -\left(\frac{\mu_1 V}{K} + \rho_1 g\right) h + p_1 = -\left(\frac{\mu_2 V}{K} + \rho_2 g\right) h + p_2 \qquad \text{at } z = 0.$$

4. This form is appropriate since the coefficients of the governing equations do not depend on z or t; k is the wavenumber and σ is the growth rate. Writing all quantities as $\hat{f}(z)\exp(\mathrm{i}kx + \sigma t)$ gives

$$\frac{\mathrm{d}^2\hat{\phi}_j}{\mathrm{d}z^2} - k^2\hat{\phi}_j = 0, \qquad \frac{\mathrm{d}^2\hat{p}_j}{\mathrm{d}z^2} - k^2\hat{p}_j = 0$$

as field equations. Solve to get $\hat{\phi}_1 = A_1 \mathrm{e}^{|k|z}$, $\hat{\phi}_2 = A_2 \mathrm{e}^{-|k|z}$, $\hat{p}_1 = B_1 \mathrm{e}^{|k|z}$, $\hat{\phi}_2 = B_2 \mathrm{e}^{-|k|z}$. The conditions at the interface give

$$|k|A_1 = -|k|A_2 = \sigma\hat{h}, \qquad -\left(\frac{\mu_1 V}{K} + \rho_1 g\right)\hat{h} + B_1 = -\left(\frac{\mu_2 V}{K} + \rho_2 g\right)\hat{h} + B_2,$$

while Darcy's law corresponds to

$$A_j = -\frac{K}{\mu_j} B_j.$$

Hence

$$\left(\frac{\mu_2 - \mu_1}{K} V + (\rho_2 - \rho_1) g\right)\hat{h} = B_2 - B_1 = \frac{\mu_1}{K} A_1 - \frac{\mu_2}{K} A_2 = (\mu_1 + \mu_2)\frac{\sigma}{K|k|}\hat{h}.$$

Hence σ is as required, although with $|k|$ to be precise.
5. These curves are straight lines. For $\rho_1 = \rho_2$, the slope is positive if $(\mu_2 - \mu_1)V > 0$, i.e. there is instability if the front moves into the more viscous fluid. For $\rho_1 \neq \rho_2$, the curves are still straight lines but the condition of instability is more complicated. The divergence at large k is not physical. The model is no longer valid beyond wavenumbers of the order of the inverse of the viscous or surface tension length scales, depending which is smaller.

6. Instability when

$$V > \frac{(\rho_1 - \rho_2)gK}{\mu_2 - \mu_1} = 10^{-7}\,\mathrm{m\,s^{-1}},$$

which is small. For a speed of 1 cm/s, $\sigma = 1$ (the gravitional term is negligible), corresponding to a time of $t = \sigma^{-1}\log 10 = 2.3\,\mathrm{s}$.

C 2.8.5 1. Conservation of mass; conservation of momentum; continuity of tangential velocity; continuity of normal velocity and definition of front velocity.
2. The Euler equations are trivially satisfied since the variables are constant. The normal velocity is zero and the fourth jump condition shows that $\overline{U}_1 = U_L$. The third is trivially satisfied since the normal is in the same direction as the base flow. The two other jump conditions reduce to

$$\rho_1\overline{U}_1 = \rho_2\overline{U}_2, \qquad P_1 + \rho_1\overline{U}_1^2 = P_2 + \rho_2\overline{U}_2^2,$$

and are clearly satisfied.
3. Small perturbations satisfy

$$\nabla\cdot\mathbf{u}_j = 0, \qquad \rho_j(\partial_t\mathbf{u}_j + \overline{U}_j\partial_x\mathbf{u}_j) = -\nabla p_j.$$

Linearize (2.30) and (2.31):

$$\mathbf{n} = (1, -\eta_y) \qquad \mathbf{w}\cdot\mathbf{n} = \partial_t\eta.$$

at $z = 0$. The jump conditions become

$$\begin{aligned} \rho_1(u_1 - \partial_t\eta) &= \rho_2(u_2 - \partial_t\eta), \\ p_1 + 2\rho_1\overline{U}_1(u_1 - \partial_t\eta) &= p_2 + 2\rho_2\overline{U}_2(u_2 - \partial_t\eta), \\ -\overline{U}_1\eta_y - v_1 &= -\overline{U}_2\eta_y - v_2, \\ u_1 - \partial_t\eta &= 0 \end{aligned}$$

at $z = 0$. These simplify to

$$u_1 = u_2 = \partial_t\eta, \quad p_1 = p_2, \quad \overline{U}_1\eta_y + v_1 = \overline{U}_2\eta_y + v_2, \qquad \text{at } z = 0.$$

4. This form is appropriate since the coefficients of the governing equations do not depend on Y or t; k is the wavenumber and σ is the growth rate. The amplitudes of the normal modes satisfy

$$\frac{\mathrm{d}\hat{u}_j}{\mathrm{d}x} + \mathrm{i}k\hat{v}_j = 0, \quad \rho_j(\sigma\hat{u}_j + \overline{U}_j\partial_x\hat{u}_j) = -\frac{\mathrm{d}\hat{p}_j}{\mathrm{d}x}, \quad \rho_j(\sigma\hat{v}_j + \overline{U}_j\partial_x\hat{v}_j) = -\mathrm{i}k\hat{p}_j.$$

5. Now take $k > 0$ and $\sigma > 0$. The divergence of the momentum equation shows that pressure is harmonic, so we have $p_1 = B_1\mathrm{e}^{kx}$ and $p_2 = B_2\mathrm{e}^{-kx}$. The momentum equations in the x-direction give

$$\hat{u}_1 = A_1\mathrm{e}^{kx}, \qquad \hat{u}_2 = A_2\mathrm{e}^{-kx} + C_2\mathrm{e}^{-\sigma x/\overline{U}_2},$$

with

$$\rho_1(\sigma + k\overline{U}_1)A_1 = -kB_1, \qquad \rho_2(\sigma - k\overline{U}_2)A_2 = kB_2,$$

The homogeneous solution for $\hat{u}_1$ vanishes because of the boundary condition as $x \to -\infty$. The continuity equation gives

$$\hat{v}_1 = \mathrm{i}A_1\mathrm{e}^{kx}, \qquad \hat{v}_2 = -\mathrm{i}A_2\mathrm{e}^{-kx} - \frac{\mathrm{i}\sigma}{kU_2}C_2\mathrm{e}^{-\sigma x/\overline{U}_2}.$$

6. The jump conditions become the four algebraic equations

$$\hat{u}_1 = \hat{u}_2 = \sigma\hat{\eta}, \quad \hat{p}_1 = \hat{p}_2, \quad \mathrm{i}k\overline{U}_1\hat{\eta} + \hat{v}_1 = \mathrm{i}k\overline{U}_2\hat{\eta} + \hat{v}_2, \qquad \text{at } z = 0.$$

The last equation can be replaced by

$$\mathrm{i}k\overline{U}_1\hat{u}_1 + \sigma\hat{v}_1 = \mathrm{i}k\overline{U}_2\hat{u}_2 + \sigma\hat{v}_2, \qquad \text{at } z = 0.$$

The quantities B_1 and B_2 can also be eliminated. The final result is three equations in three unknowns:

$$\begin{aligned}
A_1 &= A_2 + C_2, \\
\rho_1(\sigma + k\overline{U}_1)A_1 &= -\rho_2(\sigma - k\overline{U}_2)A_2, \\
\mathrm{i}k\overline{U}_1A_1 + \mathrm{i}\sigma A_1 &= \mathrm{i}k\overline{U}_2(A_2 + C_2) - \mathrm{i}\sigma\left(A_2 + \frac{\sigma}{kU_2}C_2\right).
\end{aligned}$$

This can be rewritten as the matrix system

$$\begin{pmatrix} 1 & -1 & -1 \\ r(\sigma + kU_L) & \sigma - rkU_L & 0 \\ \sigma + kU_L & \sigma - rkU_L & \sigma^2 r^{-1}k^{-1}U_L^{-1} - rkU_L \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ C_2 \end{pmatrix} = 0.$$

In order for a non-trivial solution to exist, the determinant of this system must vanish. This leads to the cubic

$$(\sigma - rkU_L)[(1+r)\sigma^2 + 2rkU_L\sigma + r(1-r)k^2U_L^2] = 0.$$

The first factor is uninteresting.
7. The solution to the quadratic is

$$\sigma = kU_L\frac{r \pm \sqrt{r(r^2 + r - 1)}}{1 + r}.$$

For $r > 1$, the square root is real. Hence the system is unstable since it has (at least) one positive root. The divergence at large k is not physical: this analysis neglects surface tension and viscosity and below the larger of these two length scales the model is no longer valid.