Solution III

D 8.10 With a broken-line profile, the Rayleigh equation reduces to $w_{zz} + \alpha^2 w = 0$. The solution in $z > 0$ is

$$w = \begin{cases} A e^{-|\alpha| z} & \text{for } z > 1 \\ B \cosh |\alpha| z + C \sinh |\alpha| z & \text{for } 0 < z < 1. \end{cases}$$

and similarly in $z < 0$. For now we just use α and put the absolute value back in at the end. The matching conditions come from continuity of the interface and of pressure:

$$\frac{w}{U - c} \quad \text{and} \quad U_z w - (U - c) w_z \qquad \text{are continuous at } z = 1.$$

(Note that $U_z = 0$ for this flow.) Since the basic flow is even, the solution w is either even (sinuous) or odd (varicose). For the sinuous solution, $C = 0$. Hence

$$\frac{A e^{-\alpha}}{-c} = \frac{C \cosh \alpha z}{1 - c}, \qquad c(-\alpha A e^{-\alpha}) = -(1 - c)(\alpha \tanh \alpha C).$$

Dividing these two equations one by the other gives $c^2 = -(1-c)^2 \tanh \alpha$, giving the dispersion relation

$$c^2 + (1 - c)^2 \tanh |\alpha| = 0.$$

Note the typo in the book. Rayleigh uses $nt + kx$, which explains the problem. This quadratic has real coefficients and a negative determinant, so it has complex conjugate roots, one of which has positive imaginary part and is hence unstable.
The varicose instability has $B = 0$, so

$$\frac{A e^{-\alpha}}{-c} = \frac{B \sinh \alpha z}{1 - c}, \qquad c(-\alpha A e^{-\alpha}) = -(1 - c)(\alpha \cosh \alpha B).$$

The same working gives

$$c^2 + (1 - c)^2 \coth |\alpha| = 0.$$

D 8.14 (i) First note that the argument of the logarithm is always positive so the streamfunction is real. Write

$$e^{\psi} = (1 + A^2)^{1/2} \cosh z + A \cos x.$$

Hence

$$e^{\psi} \nabla \psi = (-A \sin x, (1 + A^2)^{1/2} \sinh z).$$

Taking the divergence gives

$$e^{\psi} [\Delta \psi + |\nabla \psi|^2] = -A \cos x + (1 + A^2)^{1/2} \cosh z.$$

Now

$$e^{\psi}|\nabla\psi|^2 = A^2\sin^2 x + (1+A^2)\sinh^2 z.$$

and

$$e^{\psi}[-A\cos x + (1+A^2)^{1/2}\cosh z] = (1+A^2)\cosh^2 z - A^2\cos^2 x.$$

Putting this together gives

$$e^{2\psi}\Delta\psi = (1+A^2)\cosh^2 z - A^2\cos^2 x - [A^2\sin^2 x + (1+A^2)\sinh^2 z] = 1.$$

Hence $\Delta\psi = e^{-2\psi}$. A steady solution to the two-dimensional vorticity has to satisfy

$$J(\psi,\Delta\psi) = J(\psi, e^{-2\psi}) = 0,$$

since the Jacobian $J(\psi, f(\psi)) = 0$ for any function f. A contour plot of the solution for $A = 1$ is given in Figure 1.
Expand for small A in a Taylor series:

$$\begin{aligned}\psi &= \log\left[\cosh z + A\cos x + O(A^2)\right] = \log\cosh z + \log\left(1 + A\frac{\cos x}{\cosh z} + O(A^2)\right)\\ &= \log\cosh z + A\cos x\,\mathrm{sech}\, z + O(A^2).\end{aligned}$$

The first two terms can be considered as basic state streamfunction and perturbation of the flow with $\psi = \log\cosh z$, showing that the Rayleigh problem for the basic flow $U = \tanh z$ has a neutrally stable mode with wavenumber 1 and vertical structure $\mathrm{sech}\, z$.
As $A \to \infty$, have

$$\psi \sim \log A + \log\left[\cosh z + \cos x\right] + \cdots.$$

The first term is irrelevant. The second looks like a set of line vortices spaced at a distance 2π apart along the x-axis with strength -4π. This can be seen by expanding about the singularities $x = (2n+1)\pi$ and $z = 0$. This streamfunction is discussed in Lamb and can be obtained as the sum of the streamfunction for point vortices. Contours of the streamfunction are plotted in Figure 1.

8.24 Linearize the inviscid equations about the given basic state, writing the perturbation velocity as (u, v, w) and the perturbation density as ρ (subscripts are derivatives):

$$\begin{aligned}\bar\rho[u_t + \Omega u_\theta + W u_z - 2\Omega v] - \frac{V^2}{r}\rho &= -p_r,\\ \bar\rho[v_t + \Omega v_\theta + W v_z + (\Omega + V_r)u] &= -\frac{1}{r}p_\theta,\\ \bar\rho[w_t + \Omega w_\theta + W w_z + W_r u] &= -p_z,\\ \frac{1}{r}(ru)_r + \frac{1}{r}v_\theta + w_z &= 0,\\ \rho_t + \Omega\rho_\theta + W\rho_z + \bar\rho_r u &= 0.\end{aligned}$$

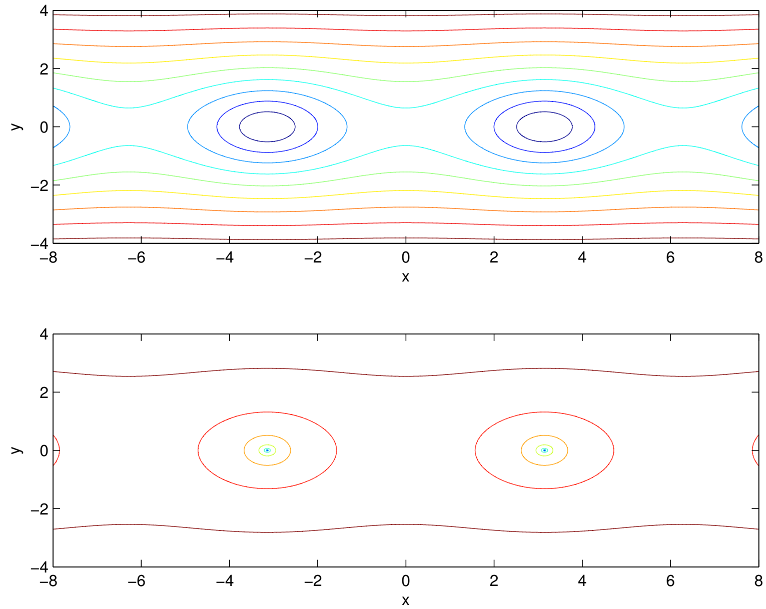

Figure 1: Top: contour plot of Stuart vortex for $A = 1$. Bottom: contour plot of limit for large A.

Here $\Omega = r^{-1}V$ is the angular velocity. Now consider the axisymmetric case and use normal modes as indicated:

$$\begin{aligned}
\bar\rho[-\mathrm{i}kc\hat u + \mathrm{i}kW\hat u - 2\Omega\hat v] - \frac{V^2}{r}\hat\rho &= -\mathrm{D}\hat p, \\
\bar\rho[-\mathrm{i}kc\hat v + \mathrm{i}kW\hat v + (\Omega + \mathrm{D}V)\hat u] &= 0, \\
\bar\rho[-\mathrm{i}kc\hat w + \mathrm{i}kW\hat w + (\mathrm{D}W)\hat u] &= -\mathrm{i}k\hat p, \\
\mathrm{D}_*\hat u + \mathrm{i}kw &= 0, \\
-\mathrm{i}kc\hat\rho + \mathrm{i}kW\hat\rho + (\mathrm{D}\bar\rho)\hat u &= 0.
\end{aligned}$$

Cancel $\hat v$ and $\hat w$:

$$\begin{aligned}
\bar\rho[-k^2(W-c)^2 + 2\Omega(\Omega + \mathrm{D}V)]\hat u - \mathrm{i}k(W-c)\frac{V^2}{r}\hat\rho &= -\mathrm{i}k(W-c)\mathrm{D}\hat p, \\
\bar\rho[-\mathrm{i}k(W-c)\mathrm{D}_* + \mathrm{i}k(\mathrm{D}W)]\hat u &= k^2\hat p, \\
\mathrm{i}k(W-c)\hat\rho + (\mathrm{D}\bar\rho)\hat u &= 0.
\end{aligned}$$

Now cancel $\hat\rho$ and substitute in $\hat u = \mathrm{i}k(W-c)F$:

$$\begin{aligned}
\bar\rho[-k^2(W-c)^2 + 2\Omega(\Omega + \mathrm{D}V)]F + (\mathrm{D}\bar\rho)\frac{V^2}{r}F &= -\mathrm{D}\hat p, \\
\bar\rho[-\mathrm{i}k(W-c)\mathrm{D}_* + \mathrm{i}k(\mathrm{D}W)]\mathrm{i}k(W-c)F &= k^2\hat p.
\end{aligned}$$

Cancel $\hat p$:

$$\mathrm{D}\{\bar\rho[(W-c)\mathrm{D}_* - (\mathrm{D}W)](W-c)F\} + \bar\rho[-k^2(W-c)^2 + 2\Omega(\Omega + \mathrm{D}V)]F + (\mathrm{D}\bar\rho)\frac{V^2}{r}F = 0.$$

Now

$$\mathrm{D}_*(W-c)F-(\mathrm{D}W)(W-c)F=(W-c)\mathrm{D}F+r^{-1}(W-c)F=(W-c)\mathrm{D}_*F,$$

which gives the required equation

$$\mathrm{D}[\bar\rho(W-c)^2\mathrm{D}_*F]-\bar\rho k^2(W-c)^2F+\Phi F=0,$$

since $\Phi=r^{-1}(\mathrm{D}\bar\rho)V^2+2\bar\rho[r^{-2}V^2+r^{-1}V(\mathrm{D}V)]$.
Notice that the condition given in the question must be wrong on dimensional grounds, since W is a velocity, while Φ has a factor of density as defined. Now write $F=(W-c)^{-1/2}G$, assuming that c has a non-zero imaginary part. Some algebra gives

$$c_i\left(\int_{R_1}^{R_2}\left\{\bar\rho(|\mathrm{D}_*G|^2+k^2|G|^2)+\frac{|G|^2}{|W-c|^2}\left[\Phi-\frac{1}{4}\bar\rho(\mathrm{D}W)^2\right]\right\}r\,\mathrm{d}r\right)=0.$$

If the flow is unstable, the square bracket must be negative somewhere, so the flow is stable if

$$\Phi\geq\frac{1}{4}\bar\rho(\mathrm{D}W)^2.$$

The subtlety in this question is that, while the analogy with the Taylor–Goldstein equation is clear, the Miles–Howard theorem derived in class used the Boussinesq approximation.

C 4.5.2 The velocity field is piecewise constant, so the perturbation velocity is irrotational. The velocity potential satisfying the boundary conditions at the walls is

$$\phi=\begin{cases}B\cosh k(y-h_2) & \text{for } 0<y<h_2\\ A\cosh k(y+h_1) & \text{for } -h_1<y<0.\end{cases}$$

The kinematic interfacial condition stays

$$\mathrm{i}k(U-c)\eta=\phi_y\qquad\text{on } y=0,$$

for both layers, while the dynamic interfacial condition becomes, from Bernoulli's' equation,

$$\rho_1[\mathrm{i}k(U_1-c)\phi_1+g\eta]-\rho_2[\mathrm{i}k(U_2-c)\phi_2+g\eta]=-k^2T\eta\qquad\text{on } y=0.$$

We hence obtain three linear equations

$$\begin{aligned}\mathrm{i}k(U_2-c)\eta &= -kB\sinh h_2,\\ \mathrm{i}k(U_1-c)\eta &= kA\sinh h_1,\\ \rho_1[\mathrm{i}k(U_1-c)A\cosh kh_1+g\eta]-\rho_2[\mathrm{i}k(U_2-c)B\cosh kh_2+g\eta] &= -k^2T\eta\end{aligned}$$

(one can eliminate η, but keeping it makes the equations cleaner). There is a non-trivial solution when the determinant

$$\begin{vmatrix} s_1 & 0 & -\mathrm{i}(U_1-c)\\ 0 & s_2 & \mathrm{i}(U_2-c)\\ \mathrm{i}k\rho_1(U_1-c)c_1 & -\mathrm{i}k\rho_2(U_2-c)c_2 & g(\rho_2-\rho_1)-k^2T\end{vmatrix}$$

vanishes, where $c_1 = \cosh kh_1$ and so on. This gives the dispersion relation

$$[g(\rho_2 - \rho_1) - k^2 T] s_1 s_2 - k\rho_2 (U_2 - c)^2 c_2 s_1 - k\rho_1 (U_1 - c)^2 c_1 s_2 = 0.$$

Simplifying gives

$$\rho_1 (U_1 - c)^2 \coth kh_1 + \rho_2 (U_2 - c)^2 \coth kh_2 + g\frac{\rho_2 - \rho_1}{k} - kT = 0.$$

One can also solve this problem using velocity, or using Rayleigh's equation and matching the total pressure.