Final

This is a take-home final. Typeset solutions should be sent to me by 11:00 am on Friday December 10. Please be concise. Try to answer the first three questions in one (possibly dense) page, in particular by quoting at the appropriate juncture results that were obtained in class. The last two questions are longer (you could aim for two pages per question).

1 (20 points) Write a one-page essay about non-dimensionalization and the principal approximations covered in this class. Aim to make the essay comprehensible to a science or engineering undergraduate with no background in continuum mechanics.

2 (20 points) Derive the equations for two-dimensional (line) thermals. Solve for the area, buoyancy and momentum in the general case. Find the solution for pure thermals. [Note: the added mass for a cylinder of mass M is M.]

3 (20 points) Find the motion of individual crests, i.e. plot some form of the relation $F(x, t, g, D) =$ constant (which you will have to solve numerically) in the case of the initial-value problem for surface gravity waves in a fluid of constant depth D.

4 (20 points) Show that small perturbations from the rest state of a compressible fluid with background density and pressure profiles $\rho_0(z)$ and $p_0(z)$ in hydrostatic balance are governed by

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \qquad \rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y}, \qquad \rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - \rho g,$$

$$\nabla\cdot\mathbf{u} + \frac{1}{\rho_0 c_s^2}\left(\frac{\partial p}{\partial t} - \rho_0 g w\right) = 0, \qquad \frac{\partial \rho}{\partial t} - \frac{\rho_0 N^2}{g} w = \frac{1}{c_s^2}\frac{\partial p}{\partial t},$$

where the buoyancy frequency for a compressible fluid is given by

$$N^2 = -g\left(\frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z} + \frac{g}{c_s^2}\right)$$

and c_s is the sound speed. Eliminate ρ and make the change of variable $\mathbf{U} = \rho_0^{1/2}\mathbf{u}$ and $P = \rho_0^{-1/2} p$ to obtain a simplified set of equations in W and P:

$$\frac{\partial}{\partial t}\left(\frac{\partial W}{\partial z} - \Gamma W\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right) P, \qquad \frac{\partial^2 W}{\partial t^2} + N^2 W = -\frac{\partial}{\partial t}\left(\frac{\partial P}{\partial z} + \Gamma P\right),$$

with $\Gamma = \frac{1}{2}(g/c_s^2 - N^2/g)$. Now assume an isothermal atmosphere with temperature T. Show that for a perfect gas the pressure and density are proportional to e^{-z/H_s} where $H_s = RT_c/g$, and that for a perfect diatomic gas with $c_s^2 = (7/5)RT_c$

$$N^2 = \frac{2}{7}\frac{g}{H_s}, \qquad \Gamma = \frac{3}{14H_s}.$$

Hence obtain the dispersion relation

$$c_s^{-2}\omega^4 - \omega^2(k^2 + l^2 + m^2 + (N/c_s)^2 + \Gamma^2) + (k^2 + l^2)N^2 = 0.$$

Plot the dispersion curves of ω/N versus kH_s. Show that there is a special mode with $W = 0$ that is non-dispersive and is trapped near the lower boundary (this is the Lamb wave). See e.g. Gill (1982).

5 (20 points) The dispersion relation for hydrostatic inertia-gravity waves is

$$\omega^2 = f^2 + \frac{N^2(k^2 + l^2)}{m^2} = F(k, l, m; f(y), N(z)),$$

where the Coriolis frequency, $f(y)$, varies with latitude and the buoyancy frequency, $N(z)$, varies with depth. Derive the ray equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{N^2 l}{m^2\omega}, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{M^2(k^2 + l^2)}{m^3\omega}, \qquad \frac{\mathrm{d}l}{\mathrm{d}t} = -\frac{\beta f}{\omega}, \qquad \frac{\mathrm{d}m}{\mathrm{d}t} = \frac{m}{N}\frac{\mathrm{d}N}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

when $f = f_0 + \beta y$. Explain why k and ω do not change in time. Find the value of y for which a ray with $(k, l) = (k_0, l_0)$ and frequency ω starting at $y = y_0$ and at the surface of the ocean ($z = 0$) hits the bottom of the ocean, in the case of an ocean with constant depth H and exponential stratification $N = N_0 e^{bz}$. Discuss the dependence on k_0 and l_0. You may wish to consider first the case $l_0 = 0$. See Garrett (2001).