Midterm Solutions

1 1. Take $\mathbf{u}\cdot$ the Euler equation and obtain

$$\mathbf{u}\cdot\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\mathbf{u}\cdot\nabla p - \rho\mathbf{u}\cdot\nabla\Phi.$$

The pressure term may be rewritten using integration by parts to give the energy equation

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\mathbf{u}^2}{2}\right) = -\rho\mathbf{u}\cdot\nabla\Phi - \nabla\cdot(\mathbf{u}p) + p\nabla\cdot\mathbf{u}.$$

The last term on the RHS is the reversible conversion to internal energy due to volume changes. For incompressible flow this term disappears since $\nabla\cdot\mathbf{u} = 0$. In this case, the internal energy equation decouples from the kinetic energy equation.

2. To obtain the Reynolds number, nondimensionalize the momentum equations with a characteristic length scale, L, characteristic velocity, U, and time scale of L/U to obtain $\mathrm{Re} = UL/\nu$, where ν is the kinematic viscosity of the fluid. This number is a ratio of inertial forces to viscous forces. When it is small, inertial forces are negligible, and viscous and pressure forces are in approximate balance. When it is large, inertia forces dominate, and the viscous term in the momentum equations may be omitted as a first approximation (outside boundary layers). The scaling used for pressure depends on whether the flow considered is for low or high Re.
The Rossby and Ekman numbers are obtained by nondimensionalizing the rotating momentum equations using the time scale f^{-1}. The pressure is nondimensionalized with ρULf. The Rossby number $\mathrm{Ro} = U/fL$ appears in front of the material derivative of velocity. When Ro is large, the earth's rotation is negligible; when Ro is small, for example when looking at very long length scales, the Coriolis force may not be ignored. The Ekman number is defined as $\mathrm{E} = \nu/fL^2$ and appears in front of the viscous term. It is a ratio of the viscous force to the Coriolis force. When E is small, the viscous forces are negligible in comparison to the Coriolis force.

3. The medium is unstratified, therefore $\mathrm{d}B/\mathrm{d}z = 0$ and $B = B_0$ is constant. Since we are looking for power-law solutions, substitute in

$$Q = qz^\lambda, \qquad M = mz^\gamma.$$

The plume equations becomes

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = \lambda qz^{\lambda-1} = \frac{2\alpha mz^\gamma}{qz^\lambda}, \qquad \frac{\mathrm{d}M}{\mathrm{d}z} = \gamma mz^{\gamma-1} = \frac{B_0 qz^\lambda}{mz^\gamma}.$$

Equating powers of z shows that $\lambda = \gamma = 1$. The equations now give

$$q = 2\alpha\frac{m}{q}, \qquad m = B_0\frac{q}{m}.$$

Solving gives

$$B = B_0, \qquad M = (2\alpha B_0^2)^{1/3}z, \qquad Q = (4\alpha^2 B_0)^{1/3}z.$$

2. The Boussinesq equations of motion (neglecting rotation) are

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0} g\hat{z} + \nu\nabla^2\mathbf{u},$$

where ρ' and p' are the perturbation density and pressure, respectively, and ρ_0 is the density of the background state. Nondimensionalize with

$$\hat{\mathbf{u}} = \mathbf{u}\frac{H}{\kappa}, \qquad \hat{\mathbf{x}} = \frac{\mathbf{x}}{H}, \qquad \hat{t} = t\frac{\kappa}{H^2}, \qquad \hat{\rho}' = \frac{\rho'}{\rho_0\beta\Delta T}, \qquad \hat{p}' = \frac{p'}{\rho_0\, u^2} = p'\frac{H^2}{\rho_0\, \kappa^2}.$$

The motivation for nondimensionalizing the perturbation density this way comes from the linearized equation of state

$$\rho = \rho_0[1 - \beta(T - T_0)]$$

and the time scale is taken to be determined by the vertical diffusion of heat. The nondimensionalized form of the Boussinesq equations of motion is

$$\frac{\kappa^2}{H^3}\frac{\mathrm{D}\hat{\mathbf{u}}}{\mathrm{D}t} = -\frac{\kappa^2}{H^3}\nabla\hat{p}' - \beta\Delta T g\hat{z}\hat{\rho}' + \frac{\nu\kappa}{H^3}\nabla^2\hat{\mathbf{u}}.$$

Rewriting this in terms of R and σ gives

$$\frac{1}{\sigma}\frac{\mathrm{D}\hat{\mathbf{u}}}{\mathrm{D}t} = -\frac{1}{\sigma}\nabla\hat{p}' - R\hat{z}\hat{\rho}' + \nabla^2\hat{\mathbf{u}},$$

where R and σ are the Rayleigh and Prandtl number, respectively. From the equation of state the units of β are 1/[T]. The flow will be turbulent for large R because it will have a large driving term, leading to vigorous overturning motions (R is essentially a Reynolds number with the velocity scale determined by the buoyancy forcing).

3 (i) The length scale of the boundary layer thickness is given as δ, and the bottom plate must be at a temperature $T_0 + \Delta T$. The temperature difference δT between the lower plate and the interior is therefore

$$\delta T = T_0 + \Delta T - (T_0 + \frac{1}{2}\Delta T) = \frac{1}{2}\Delta T.$$

Using this temperature difference and the boundary layer thickness as the length scale, the local Rayleigh number is

$$R_\delta = \frac{g\beta\Delta T\delta^3}{2\kappa\nu} = \frac{g'\delta^3}{\kappa\nu}.$$

(ii) Since the local Rayleigh number is approximately 1800 for a blob about to leave the boundary layer,

$$\frac{g'\delta^3}{\kappa\nu} = 1800$$

and hence

$$\frac{\delta}{H} = \left(\frac{1800\,\kappa\,\nu}{g'H^3}\right)^{1/3}.$$

(iii) When the thermal is about to leave the boundary layer, its diameter is on the order of δ. Since it is assumed to be spherical, an appropriate volume at time of release is $V_0 = \delta^3$. It is assumed that the blob has no initial velocity, and consequently no momentum, at time of release so that $M_0 = 0$. The specific buoyancy is given by volume times buoyancy acceleration g', where

$$g' = \frac{\rho_e - \rho}{\rho_0} g$$

with the usual meanings for the densities. In terms of temperature, use

$$\frac{\rho_e - \rho}{\rho_0} = \beta(T - T_0)$$

to obtain $g' = \beta(T - T_0)g$. The temperature difference between the bottom plate and the interior is one half the total temperature difference so

$$g' = \frac{1}{2} g \beta \Delta T$$

and the initial buoyancy may be taken as $B_0 = g'\delta^3$.

(iv) The thermal equations are

$$\frac{\mathrm{d}V}{\mathrm{d}z} = 3\alpha_T V^{2/3}, \qquad \frac{\mathrm{d}M}{\mathrm{d}z} = \frac{2}{3}\frac{BV}{M}, \qquad \frac{\mathrm{d}B}{\mathrm{d}z} = -N^2 V$$

where α_T is the thermal entrainment coefficient and N^2 is the buoyancy frequency. Because we consider only rise in the interior, where the temperature is constant, $N^2 = 0$ and $B = B_0 = g'\delta^3$ is constant. Separate variables in the volume equation:

$$\frac{\mathrm{d}V}{V^{2/3}} = 3\alpha_T \mathrm{d}z$$

and integrate: from $z = \delta$ to $z = H - \delta$:

$$3[V^{1/3} - V_0^{1/3}] = 3\alpha_T (z - \delta).$$

At $z = \delta$, this gives

$$V = [\alpha_T (H - 2\delta) + \delta]^3 \approx (\alpha_T H)^3,$$

since $\delta \ll H$. The thermal is expanding from its original size of δ^3 through entrainment of fluid from the ambient. The momentum equation is

$$\frac{\mathrm{d}M}{\mathrm{d}z} = \frac{2}{3}\frac{B_0}{M}[\alpha_T (z - \delta) + \delta]^3.$$

Separating variables and integrate:

$$\frac{1}{2}M^2 = \frac{2}{3}B_0 \frac{1}{4\alpha_T}([\alpha_T (z - \delta) + \delta]^4 - \delta^4).$$

The final value of momentum is

$$M = \left[\frac{B_0}{3\alpha_T}([\alpha_T (H - 2\delta) + \delta]^4 - \delta^4)\right]^{1/2} \approx \left[\frac{B_0 \alpha_T^3 H^4}{3}\right]^{1/2}.$$

The final temperature of the blob is the same as when it starts, since g' has not changed. This seems reasonable: the blob moves rapidly through the boundary layer and diffusion is negligible on this advective time scale. Fluid is entrained so the volume increases. The momentum also increases as the blob rises and potential energy is converted to kinetic energy.