Solution I

1. Consider a tensor of arbitrary order, $F(\mathbf{x},t)$, integrated over the region $V(t)$, which may be either a fixed or material volume. The time derivative of this tensor is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} F\,\mathrm{d}V,$$

where $\mathrm{d}/\mathrm{d}t$ emphasizes the fact that F remains solely a function of time after integration in space. Exploring the 1D case, with boundaries at $x=a(t)$ and $x=b(t)$, one can use Leibniz's theorem, which shows how to differentiate an integral whose integrand F as well as the limits of integration are functions of the variable with respect to which the integration occurs. The time derivative of the integral becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} F(x,t)\,\mathrm{d}x = \int_a^b \frac{\partial F}{\partial t}\,\mathrm{d}x + \left(\frac{\mathrm{d}b}{\mathrm{d}t}F(b,t) - \frac{\mathrm{d}a}{\mathrm{d}t}F(a,t)\right).$$

The last two terms on the right-hand side represent the gain of F at the outer boundary moving at a rate $\mathrm{d}b/\mathrm{d}t$, and the loss of F at the inner boundary at a rate $\mathrm{d}a/\mathrm{d}t$, respectively. These may be combined to a more general integral, representing the outward flux through an area element $\mathrm{d}\mathbf{A}$. Rewriting the above equation, and generalizing Leibniz's theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} F(\mathbf{x},t)\,\mathrm{d}V = \int_{V(t)} \frac{\partial F}{\partial t}\,\mathrm{d}V + \int_{A(t)} F\,\mathrm{d}\mathbf{A}\cdot\mathbf{u}_A,$$

where $\mathbf{u}_A$ is the velocity of the boundary and $A(t)$ is the surface of $V(t)$.
For a material volume, $\mathbb{V}(t)$, the surfaces move with the fluid, and consequently $\mathbf{u}_A = \mathbf{u}$. The above equation may be re-written as the Reynolds transport theorem, following a material volume:

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathbb{V}} F(\mathbf{x},t)\,\mathrm{d}\mathbb{V} = \int_{\mathbb{V}} \frac{\partial F}{\partial t} d\mathbb{V} + \int_A F\,\mathrm{d}\mathbf{A}\cdot\mathbf{u}_A.$$

Another form of the transport theorem may be obtained by manipulating the above via Gauss's theorem, and defining a new function f, where $F=\rho f$. Transforming the surface integral into a volume integral, and using index notation for simplicity,

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathbb{V}} \rho f\,\mathrm{d}\mathbb{V} = \int_{\mathbb{V}} \left[\frac{\partial F}{\partial t} + \frac{\partial}{\partial x_j}(\rho f u_j)\right]\mathrm{d}\mathbb{V}.$$

By virtue of the product rule, the RHS becomes

$$= \int_{\mathbb{V}} \left[\rho\frac{\partial f}{\partial t} + f\frac{\partial \rho}{\partial t} + f\frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j \frac{\partial f}{\partial x_j}\right]\mathrm{d}\mathbb{V}.$$

The continuity equation gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

so that

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathbb{V}} \rho f \, \mathrm{d}\mathbb{V} = \int_{\mathbb{V}} \left[\rho \frac{\partial f}{\partial t} + \rho u_j \frac{\partial f}{\partial x_j}\right] d\mathbb{V} = \int_{\mathbb{V}} \rho \frac{\mathrm{D}f}{\mathrm{D}t} \, \mathrm{d}\mathbb{V}.$$

SOURCE: Kundu et al., **Fluid Mechanics**, ed. 4, pgs. 82–84.

2. Upon nondimensionalizing the momentum equations with a characteristic length scale, L, and characteristic velocity, U, the non-dimensional Reynolds number is obtained:

$$Re = \frac{UL}{\nu},$$

where ν is the kinematic viscosity of the fluid. This quantity is a ratio of inertia forces to viscous forces; if small, inertia forces are negligible, and viscous and pressure forces are in approximate balance in steady state (e.g. creeping flow around a sphere). When Re is large, inertia forces dominate, and the viscous term in the momentum equations may be omitted as a first approximation, outside of the boundary layer.

The Rossby and Ekman numbers are obtained by nondimensionalizing the rotating momentum equations. The Rossby number is given by

$$Ro = \frac{U^2/L}{fU} = \frac{U}{fL}$$

and represents the ratio of the nonlinear advection terms to the Coriolis force. When Ro is large, the earth's rotation is negligible, as opposed to the nonlinear acceleration terms. When Ro is small, for example when looking at very long length scales, the Coriolis force may not be ignored, but the neglect of nonlinear terms is justified for the flow.

The Ekman number is defined as

$$E = \frac{\nu}{fL^2}$$

and is a ratio of the viscous force to the Coriolis force. Likewise, when the flow in question is over a long characteristic length scale, and consequently E is very small, the viscous forces are negligible in comparison to the force created by earth's rotation.

(i) For mantle convection, the following approximate values were found:

$$\nu \approx 3 \times 10^{17} \text{ m}^2/\text{s}, \quad U \approx 1.5 \times 10^{-9} \text{ m/s}, \quad D \approx 2.9 \times 10^6 \text{ m}$$

The Coriolis frequency is defined as $f = 2\Omega \sin\theta$, where Ω is the earth's rate of rotation, equal to 0.73×10^{-4} s^{-1}, and θ is the latitude. Here, the latitude is assumed to be 90°N, giving $f = 1.46 \times 10^{-4}$ s^{-1}. Therefore, the three dimensionless parameters are:

$$Re = 1.45 \times 10^{-20}, \quad Ro = 3.54 \times 10^{-12}, \quad E = 2.44 \times 10^{8}$$

These values make sense intuitively, since mantle convection is a slow creeping motion dominated by viscous effects.

SOURCE: Lowrie, William. **Fundamentals of Geophysics**, pg. 328.

(ii) For the Great Red Spot of Jupiter, the following approximate values were found:

$$\nu \approx 1 \times 10^6 \text{ m}^2/\text{s}, \quad U \approx 2 \text{ m/s}, \quad D \approx 1.3 \times 10^7 \text{ m}, \quad f \approx 1.3 \times 10^{-4} \text{ s}^{-1}$$

These correspond to a Reynolds, Rossby, and Ekman number of:

$$Re = 26, \quad Ro = 1.18 \times 10^{-3}, \quad E = 4.55 \times 10^{-5}$$

In this case, it is expected that viscous forces do not dominate the flow, but rather the Coriolis terms. The result for *Re* seems small. These may not be the best values to take.

SOURCE: Stone, P.H. et al., *Concerning the existence of Taylor columns in atmospheres*, Journal of Royal Meter. So., 1968, pgs. 578-579.

(iii) The outflow from the Point Loma wastewater treatment plant has the following approximate values:

$$\nu \approx 1 \times 10^{-6} \text{ m}^2/\text{s}, \quad U \approx 0.731 \text{ m/s}, \quad D = 3.6576 \text{ m}$$

Located at a latitude of 33°N, the Coriolis frequency is $f = 7.95 \times 10^{-5}$ s^{-1}. These values give the following nondimensional parameters:

$$Re = 2.67 \times 10^6, \quad Ro = 2.51 \times 10^3, \quad E = 9.40 \times 10^{-4}$$

In this case, *Re* is large, and one would therefore expect a turbulent outflow from the pipe. Furthermore, the Rossby number is large, and consequently the nonlinear acceleration terms cannot be ignored, as is expected for turbulent flow.

SOURCE: City of San Diego Facilities website, Point Loma Wastewater Treatment Plant.
`http://www.sandiego.gov/mwwd/facilities/ptloma.shtml`

3. The vertical momentum equation for a static fluid of density ρ is

$$0 = -\frac{\mathrm{d}p}{\mathrm{d}z} - \rho g.$$

This equation may be readily integrated in z to obtain

$$P = -\rho g z,$$

where the atmospheric pressure, P_0, has been taken to be 0 at $z = 0$. Now consider a body with boundary S that is at least partially submerged in the fluid. The fluid exerts a force $d\mathbf{F} = -P\,\hat{\mathbf{n}}\,dS$ on an area element dS on the body, where $\hat{\mathbf{n}}$ is the unit outward vector normal to the surface. The net forces in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions are 0 by symmetry, so that the total force in the z-direction is:

$$F_z = -\int_S P\,\mathrm{d}S.$$

Substituting for P,

$$F_z = \rho g \int_S z\,\mathrm{d}S.$$

By virtue of the divergence theorem, the force becomes

$$F_z = \rho g \int_V \frac{\partial z}{\partial z} \mathrm{d}V = \rho g \int_V \mathrm{d}V = \rho g V = mg.$$

Therefore, the force acting on the solid object immersed in the fluid is equal to the weight of the displaced fluid. Note that the hydrostatic equation is used on the boundary to obtain F_z. The transformation from surface to volume integral is a mathematical procedure and the physical contents of the interior of the volume are irrelevant.

SOURCE: Physics Wiki, Archimedes' Principle

4. The fluid in the tank is in solid-body rotation, and therefore the inviscid Euler equations in the radial and vertical directions may be used in cylindrical coordinates, and simplifies to

$$-\rho \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r}, \qquad 0 = -\frac{\partial p}{\partial z} - \rho g.$$

The pressure differences between two neighboring points must be:

$$\mathrm{d}p = \frac{\partial p}{\partial r} \mathrm{d}r + \frac{\partial p}{\partial z} \mathrm{d}z = \rho \Omega^2 r \, \mathrm{d}r - \rho g \, \mathrm{d}z,$$

where Ωr has been substituted for u_θ. This may be integrated between arbitrary points 1 and 2 so that

$$P_2 - P_1 = \frac{1}{2} \rho \Omega^2 (r_2^2 - r_1^2) - \rho g (z_2 - z_1)$$

Surfaces of constant pressure, and thus an outline of the surface, occur at

$$z_2 - z_1 = \frac{1}{2} \frac{\Omega^2}{g} (r_2^2 - r_1^2),$$

which are paraboloids of revolution. If at height $z_1 = 0$ (taken at the bottom of the tank), the radius is $r_1 = 0$, since all the fluid is pushed up against the side walls, then the above equation becomes

$$z_2 = \frac{1}{2} \frac{\Omega^2}{g} r_2^2.$$

Working in 2D, the area under the parabola must equal the area of fluid in the tank, which is 2HL. Therefore,

$$2 \int_0^L \frac{1}{2} \frac{\Omega^2}{g} x^2 \, \mathrm{d}x = 2HL$$

Integrating the LHS,

$$\frac{1}{6} \frac{\Omega^2}{g} L^3 = HL$$

Solving for Ω_c,

$$\Omega_c = \sqrt{\frac{6gH}{L^2}}$$

If Ω is increased above this value, the minimum point on the parabola of the free surface (i.e. line of constant pressure) will have to drop below z = 0, so that at z = 0, the bottom of the tank is dry.