Solution II

1 The volume, momentum and buoyancy flux (per unit length) are given by

$$Q = 2\int_0^\infty w\,\mathrm{d}x, \qquad M = 2\int_0^\infty w^2\,\mathrm{d}x, \qquad B = 2\int_0^\infty g' w\,\mathrm{d}x,$$

where

$$g' = \frac{\rho_e - \rho}{\rho_0} g$$

is the specific gravity, as in class, with ρ_e the density in the ambient, ρ the density inside the plume, and ρ_0 the reference density.

Exactly the same approach as in class works, with the obvious change in the continuity equation: $r^{-1}\partial_r(ru)$ goes to $\partial_x(u)$. We obtain the three following equations, which are just those for the axisymmetric plume with b (the half-width of the plume) taken out in the appropriate place:

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = 2u_e, \qquad \frac{\mathrm{d}M}{\mathrm{d}z} = 2\int_0^b g'\,\mathrm{d}x, \qquad \frac{\mathrm{d}B}{\mathrm{d}z} = -N^2(z)Q.$$

Now use top hat velocity profiles. Then the entrainment assumption states that the entrainment velocity is proportional to w, so that $u_e = \alpha w$, where α is the entrainment coefficient. On dimensional grounds, $w = MQ^{-1}$ and $b = Q^2M^{-1}$. Then, using the same arguments as for the axisymmetric case, we can obtain a closed set of equations in the form

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = \frac{2\alpha M}{Q}, \qquad \frac{\mathrm{d}M}{\mathrm{d}z} = \frac{BQ}{M}, \qquad \frac{\mathrm{d}B}{\mathrm{d}z} = -N^2(z)Q.$$

Notice that the only difference is in the first equation. The pure plume solution in an unstratified ambient is particularly simple since both Q and M are linear in z.

2 The equations for an axisymmetric pure plume are

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = 2\alpha M^{1/2}, \qquad \frac{\mathrm{d}M}{\mathrm{d}z} = \frac{BQ}{M}, \qquad \frac{\mathrm{d}B}{\mathrm{d}z} = -N^2Q$$

with initial conditions $Q_0 = M_0 = 0$ and $B_0 = 1$. Following the non-dimensionalization of CW98, we obtain

$$\hat{B} = \frac{B}{B_s}, \quad \hat{Q} = \frac{Q}{(2\alpha)^{4/3}B_s^{1/3}H_p^{5/3}}, \quad \hat{M} = \frac{M}{(2\alpha)^{2/3}B_s^{2/3}H_p^{4/3}}, \quad \hat{z} = \frac{z}{H_p}, \quad \hat{N} = \frac{N}{N_s},$$

where H_p is the characteristic length scale of the plume height of rise. Then the plume equations take the universal form

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z} = \hat{M}^{1/2}, \qquad \frac{\mathrm{d}\hat{M}}{\mathrm{d}z} = \frac{\hat{B}\hat{Q}}{\hat{M}}, \qquad \frac{\mathrm{d}\hat{B}}{\mathrm{d}z} = -\hat{N}^2\hat{Q}.$$

A plot of the integrated ODEs against height is given in Figure 1. The MATLAB program shows how to produce this graph in a number of ways. Using the plume equations is straightforward, but BM/Q is not defined numerically at $z=0$. The simplest approach is to give small positive values to Q and M near the origin – this works. Alternatively one can start the integration at the small value $z=\varepsilon$, using the fact that near the origin the plume does not feel the effect of stratification, so that $Q\sim q_0 z^{5/3}$ and $M\sim m_0 z^{4/3}$. Mathematically it is more elegant to use the variables $q=z^{-2/3}Q$ and $m=z^{-1/3}M$. These variables have finite derivatives q_0 and m_0 at the origin. One can extend this to deal with $N=z^a$ with negative a by writing $B=1+bz^{a+5/3}$. This is not actually needed in CW98 since they start at $z=\lambda^{-1}$ (see below).
Physically, Q increases with height due to entrainment, while buoyancy decreases with height due to the entrainment of relatively dense ambient fluid. When B goes to 0, the momentum decreases so that the plume will eventually stop at a finite height $z_m=2.57\ldots$. Buoyancy has units L^4T^{-3} and the buoyancy frequency, N, has units T^{-1}, so the maximum height becomes $z_m B^{1/4}N^{-3/4}$.

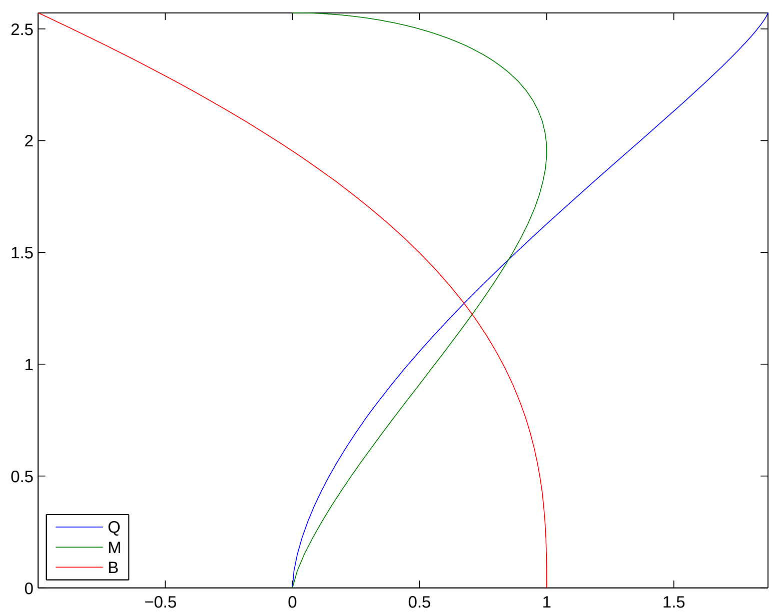

Figure 1: Volume, momentum and buoyancy fluxes vs. height for a pure plume in a stratified ambient with $N=1$.

For $a>0$, the stratification becomes stronger with height, so that the plume will reach even less high than for the case $a=0$. When $a<0$, stratification becomes increasingly weaker with height, so the plume may or may not be bounded. CW define the parameter $\lambda=H_p/z_s$ and have the plume start at $z=\lambda^{-1}$. Then λ represents the ratio of the scale height rise of a plume in a uniformly stratified environment to the scale height of the environmental stratification. In our case, the plume starts at the origin and N^2 is unbounded there for $a<0$, and it is not clear this is a good physical model.
Here is what CW say in their abstract.

> In the case $a>0$, the stratification becomes progressively stronger with height, and so plumes are always confined within a finite distance above the origin. Furthermore, the

non-dimensional height of rise h decreases with λ. In contrast, in the case $a < 0$, the stratification becomes progressively weaker with height, and so the non-dimensional plume height increases monotonically with λ. For slowly decaying stratification, $\beta > -8/3$, the motion is confined within a finite distance above the source. However, for each value of a with $a < -8/3$, there is a critical value $\lambda_c(a)$ such that for $\lambda < \lambda_c$ a plume is confined to a region near the source while for $\lambda \geq \lambda_C$ the motion is unbounded. [...]

3 First consider the case with no stratification. A pure plume starts to rise with very small length-scale. As it rises its radius becomes larger (and its velocity also changes). Eventually the Rossby number based on its width and entrainment velocity will reach $O(1)$. Now we can use the pure plume result to find

$$\mathrm{Ro} = \frac{u}{bf} \sim \frac{B^{1/3}}{z^{4/3} f}.$$

This gives an estimate $H \sim B^{1/4} f^{-3/4}$. At this height rotation will act to suppress vertical shear and the plume will presumably start to spread out as some kind of thin vortex.

With stratification, the Prandtl ratio N/f enters the picture. If $N \ll f$, then the plume will not have reached an equilibrium level by the time it starts to feel rotation. If N and f are comparable, the plume may reach its neutral level and start to spread out as a buoyancy current before it feels rotation. We have not discussed this spreading-out phase, but by conservation of volume, the plume must keep spreading, so eventually its size will become large enough to feel rotation, although the behavior of the spreading velocity will also be relevant.

See Bush & Woods (1999).