Solution III

1. Consider an element of arc $\mathrm{d}s$ on the free surface: the surface tension and inner pressure p must be balanced by the pressure outside of the interface, p_a. The balance of forces perpendicular to the arc requires

$$-p_a \,\mathrm{d}s + p\,\mathrm{d}s + \sigma\,\mathrm{d}\theta = 0.$$

The pressure difference is therefore given by

$$p_a - p = \sigma\frac{\mathrm{d}\theta}{\mathrm{d}s} = \sigma\frac{\partial^2\eta}{\partial x^2}$$

in two dimensions, where the curvature κ is defined as

$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

and where the free surface is parameterized by $x = x(t)$, $y = y(t)$. For a two-dimensional curve with $y = \eta(x)$, the curvature becomes

$$\kappa = \frac{\partial^2\eta/\partial x^2}{[1 + (\partial\eta/\partial x)^2]^{3/2}} \approx \frac{\partial^2\eta}{\partial x^2}$$

for small slopes. If the atmospheric pressure is taken to be 0, the linearized dynamic condition at the surface is

$$p = -\sigma\frac{\partial^2\eta}{\partial x^2} \qquad \text{at } z = 0.$$

This may be combined with the linearized Bernoulli equation

$$\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + gz = 0$$

to give

$$\frac{\partial\phi}{\partial t} = \frac{\sigma}{\rho}\frac{\partial^2\eta}{\partial x^2} - g\eta.$$

The solution to the Laplace equation is as in class, namely

$$\phi = \frac{a\omega}{k}\frac{\cosh k(z+H)}{\sinh kH}\sin(kx - \omega t),$$

with free surface,

$$\eta = a\cos(kx - \omega t).$$

The new dynamic condition at $z=0$ gives the dispersion relation

$$\omega^2 = k\left(g + \frac{\sigma k^2}{\rho}\right)\tanh kH.$$

For the three-dimensional case, the k^2 term is replaced by k^2+l^2. The group velocity c_g is given by $\partial\omega/\partial k$, so

$$c_g = \frac{(g/k)^{1/2}}{2}\frac{[1+3T(kH)^2]\tanh kH + kH[1+T(kH)^3]\mathrm{sech}^2 kH}{\{[1+T(kH)^2]\tanh kH\}^{1/2}},$$

where $T=\sigma/\rho H^2$ is a non-dimensional version of surface tension. In the limit of small kH, $c_g \approx \sqrt{gH}$, the standard shallow water case (assuming T is not ridiculously large). For large kH and T not very small, $c_g \approx (3/2)\sqrt{\sigma k/\rho}$, corresponding to capillary waves on deep water.

2 Using the Lagrangian description for fluid motion, consider a fluid particle at $(x_0+\lambda, z_0+\zeta)$ with mean position (x_0, z_0). Then

$$u = \frac{\partial\lambda}{\partial t}, \qquad w = \frac{\partial\zeta}{\partial t},$$

while the velocities are found by taking the gradient of the velocity potential above:

$$u = a\omega\frac{\cosh k(z+H)}{\sinh kH}\cos(kx-\omega t), \qquad w = a\omega\frac{\sinh k(z+H)}{\sinh kH}\sin(kx-\omega t).$$

For small-amplitude waves, the motion of the particle is small and consequently the velocity of a particle along its path is approximately equal to the velocity at the mean position (x_0, z_0), so that

$$\frac{\partial\lambda}{\partial t} = a\omega\frac{\cosh k(z_0+H)}{\sinh kH}\cos(kx_0-\omega t), \qquad \frac{\partial\zeta}{\partial t} = a\omega\frac{\sinh k(z_0+H)}{\sinh kH}\sin(kx_0-\omega t).$$

Integrating in time gives

$$\lambda = -a\frac{\cosh k(z_0+H)}{\sinh kH}\sin(kx_0-\omega t), \qquad \zeta = a\frac{\sinh k(z_0+H)}{\sinh kH}\cos(kx_0-\omega t)$$

(the constant of integration is irrelevant). Using the identity $\sin^2 x + \cos^2 x = 1$, the particle path may be written as

$$\frac{\lambda^2}{[\cosh k(z_0+H)/\sinh kH]^2} + \frac{\zeta^2}{[\sinh k(z_0+H)/\sinh kH]^2} = a^2;$$

these are ellipses. In the deep water case,

$$\frac{\cosh k(z_0+H)}{\sinh kH} \simeq \frac{\sinh k(z_0+H)}{\sinh kH} \simeq e^{kz_0},$$

that the particle orbits are

$$\lambda = -ae^{kz_0}\sin(kx_0-\omega t), \qquad \zeta = ae^{kz_0}\cos(kx_0-\omega t).$$

The equation for the particle paths becomes

$$\zeta^2 + \lambda^2 = a^2 e^{2kz_0}.$$

i.e. circles whose radius is equal to the amplitude a at the surface and decreases exponentially with depth. For shallow water, the approximations

$$\cosh k(z_0 + H) \simeq 1, \qquad \sinh k(z_0 + H) \simeq k(z_0 + H), \qquad \sinh kH \simeq kH$$

lead to

$$\lambda = -\frac{a}{kH} \sin(kx_0 - \omega t), \qquad \zeta = a\left(1 + \frac{z_0}{H}\right)\cos(kx_0 - \omega t).$$

These are thin ellipses described by

$$(\lambda kH)^2 + \frac{\zeta^2}{(1 + z_0/H)^2} = a^2.$$

In this limit, the semi-major axis, a/kH, is depth-independent, and the semi-minor axis decreases linearly to 0 at the bottom boundary. Therefore, the particle orbits are ellipses that get progressively thinner for particle near the bottom.

3 We need to solve Laplace's equation with the usual boundary conditions in $x > 0$. However we are now imposing a velocity $u = \partial\phi/\partial x$ on $x = 0$ (for inviscid flow we can only impose the normal velocity). We seek a separated solution in the form $\phi = X(x)Z(z)\mathrm{e}^{-\mathrm{i}\omega t}$. Laplace's equation gives

$$X'' - \kappa^2 X, \qquad Z'' + \kappa^2 Z = 0.$$

The linearized boundary conditions for Z are $Z' = 0$ and $z = -H$ and $-\omega^2 Z + gZ' = 0$ at $z = 0$. We take $Z = \cos\kappa(z + H)$ to satisfy the first of these. The second then gives

$$\tan\kappa H = -\frac{\omega^2}{g\kappa}.$$

Plotting the two sides of this equation shows that there are infinitely positive roots κ_n. These correspond to modes Z_n that decay for large x, i.e. trapped near the wall. However, there is also a mode with imaginary $\kappa = \mathrm{i}k$ with $Z = \cosh k(z + H)$ and

$$\tanh kH = \frac{\omega^2}{gk}.$$

Here we must take the mode with positive $k = k_0$ to obtain waves that propagate away from the wall. Hence

$$\phi = A_0 \mathrm{e}^{\mathrm{i}(k_0 x - \omega t)} Z_0(z) + \sum_{n=1}^{\infty} A_n \mathrm{e}^{-\kappa_n x - \mathrm{i}\omega t} Z_n(z).$$

The coefficients are found by matching the velocity at the wall to the forcing:

$$u = \mathrm{i}k_0 A_0 Z_0(z) - \sum_{n=1}^{\infty} \kappa_n A_n Z_n(z).$$

The vertical modes come from a Sturm–Liouville problem, so we may take them to be orthonormal. Then the boundary condition may be multiplied by Z_m^* and integrated to give

$$\mathrm{i}k_0 A_0 = \int_{-H}^{0} u(z) Z_0(z)\,\mathrm{d}z, \qquad -\kappa_n A_n = \int_{-H}^{0} u(z) Z_n(z)\,\mathrm{d}z.$$

Only the portion of u that projects onto the mode Z_0 (essentially the lowest mode) gives a propagating response.