Solution IV

1 See Cahn (1945).

2 In class, we saw that an internal wave will reflect forward off a slope with angle less than the critical angle θ, but back off a slope with angles greater than the critical angle. Hence an internal wave (packet) will just be reflected off a wedge with a large opening angle. If the angle is small, it will reflect into the wedge. The wave reflects specularly off the horizontal boundary, so it continues into the wedge. Hence energy appears to propagate into the wedge. The analysis in Wunsch (1968) finds the appropriate mathematical solution, which has a singularity at the tip of the wedge. In practise, viscous effects will become important as the amplitude of the wave grows. See also SGLS (2004).

3 This can be found in a number of courses, e.g. Kundu & Cohen or Pedlosky. I will save paper and disk space by not repeating the analysis.

4 *The most popular sources were Chapman & Malanotte-Rizzoli and Kundu & Cohen. Neither treatment was very clear about the range of* ω. We can follow the treatment in class with the following differences. The horizontal momentum equations become

$$u_t - fv = -\rho_0^{-1} p_x, \qquad v_t + fu = -\rho_0^{-1} p_y.$$

Solving for u and v gives

$$(\partial_{tt}^2 + f^2)u = -\rho_0^{-1} p_{xt} - f\rho_0^{-1} p_y, \qquad (\partial_{tt}^2 + f^2)v = -\rho_0^{-1} p_{yt} + f\rho_0^{-1} p_x.$$

Hence

$$(\partial_{tt}^2 + f^2)w_z = \rho_0^{-1}\nabla_H^2 p_t.$$

Take ∂_z and eliminate p using the same equation as in class:

$$(\partial_{tt}^2 + N^2)w = -\rho_0^{-1} p_{tz}.$$

Making the Boussinesq approximation gives

$$\partial_{tt}^2 \nabla^2 w + f^2 w_{zz} + N^2 \nabla_h^2 w = 0.$$

Substituting in the plane wave solution $e^{i(kx+ly+mz-\omega t)}$ gives the dispersion relation

$$\omega^2 = \frac{N^2(k^2+l^2) + f^2 m^2}{k^2+l^2+m^2} = N^2\cos^2\theta + f^2\sin^2\theta.$$

In geophysical situations, $|f| < N$, so inertia-gravity waves occupy the frequency band $f^2 \leq \omega^2 \leq N^2$.
Another approach that can be useful is as follows. Seek a plane wave solution straight away. The governing equations then give five homogeneous linear equations:

$$\begin{pmatrix} -\mathrm{i}\omega & -f & 0 & \mathrm{i}k\rho_0^{-1} & 0 \\ f & -\mathrm{i}\omega & 0 & \mathrm{i}l\rho_0^{-1} & 0 \\ 0 & 0 & -\mathrm{i}\omega & \mathrm{i}k\rho_0^{-1} & g \\ \mathrm{i}k & \mathrm{i}l & \mathrm{i}m & 0 & 0 \\ 0 & 0 & \mathrm{d}\rho_0/\mathrm{d}z & -\mathrm{i}\omega & \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ p \\ \rho \end{pmatrix} = \mathbf{0}.$$

The determinant of the equation must vanish. Taking f and N to be constant gives the dispersion relation.