Thermals

Introduction Thermals correspond to an instantaneous finite release of buoyancy. First discussed by Morton, Taylor & Turner (1956; MTT). Follow their treatment. This derivation is essentially that of MTT, and it's useful to go back and compare it to the derivation of the plume equations.

Variables Observations show that thermals rise from the ground and are approximately spherical in shape. We will derive a system analogous to that for plumes. We use the following variables: V for volume (length), P for specific momentum (length^4/time) and specific buoyancy (length^4/time^2). Using a top-hat profile, these can be related to an effective radius b, a characteristic velocity w and a reduced gravity $g' = g(\rho - \rho_e)\rho_0$, where ρ is the density of the plume, $\rho_e(z)$ is the density in the ambient and ρ_0 is a reference density (we are using the Boussinesq approximation). This gives $V = b^3$, $P = b^3 w$ and $B = b^3 g'$ (note that these definitions are not unique).
The three governing equation come from conservation of mass, Newton II and the first law. In these three cases, we look at the time rate of change of the total, integrated quantity in the thermal. We then replace $\mathrm{d}/\mathrm{d}t$ by $w\mathrm{d}/\mathrm{d}z$.

Volume Since we are considering a Boussinesq system, we can replace conservation of mass by conservation of volume. In the plume equations, the info at the edge of the plume depended on the horizontal velocity at the edge of the plume, u_e, which was related to the characteristic vertical velocity in the plume, w, by the entrainment assumption by $w = \alpha u_e$, where α is the entrainment parameter. Here we argue that the volume of the thermal can change as fluid flows through the boundary. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}V = \int_S u_n \, \mathrm{d}S.$$

Now the quantity we care about in the thermal is its characteristic vertical velocity w, so we will say that the inflow over the boundary is proportional to the area times w times an entrainment coefficient α_T, which is not the same as for the plume case. Observations indicate $\alpha_T \sim 0.25$. Using the change of variable, we find

$$w\frac{\mathrm{d}}{\mathrm{d}z}(4\pi/3)b^3 = 4\pi b^2 \alpha_T w.$$

In terms of the specific variables, this is

$$\frac{\mathrm{d}V}{\mathrm{d}z} = 3\alpha_T V^{2/3}.$$

It is interesting that w cancels and this equation involves only V.

Momentum The rate of change of momentum is given by the force on the thermal. The latter has two terms: gravity and the buoyancy force, i.e. the mass of displaced fluid. We can use the reference density ρ_0 in the definition of momentum, but the critical point is that we need to consider not just the mass of the thermal, but also the mass of fluid it displaces, known as the added mass.[1] For a sphere, the added mass is $M/2$. Hence we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(2\pi/3 + 4\pi/3)\rho_0 b^3 w = g(\rho_e - \rho)(4\pi/3)b^3.$$

This can be simplified to give

$$\frac{\mathrm{d}}{\mathrm{d}z}(b^3 w) = \frac{2}{3} g \frac{\rho_e - \rho}{\rho_0}\frac{b^3}{w}.$$

Hence we obtain the final form

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{2}{3}\frac{BV}{P}.$$

Buoyancy The first law of thermodynamics is really the heat equation, which in this context is the advection equation for density, since we are neglecting molecular viscosity. Physically, this means that density can only change by a flux in from the boundary, which is most appropriately formulated in terms of density departures from the reference density:

$$\frac{\mathrm{d}}{\mathrm{d}t}(4\pi/3)b^3(\rho - \rho_0) = 4\pi\alpha_T w(\rho_e - \rho_0),$$

using the same entrainment hypothesis as before. Now use

$$w\frac{\mathrm{d}}{\mathrm{d}z}(4\pi/3)b^3(\rho_0 - \rho_e) = (\rho_0 - \rho_e)4\pi\alpha_T w - (4\pi/3)b^3 w\frac{\mathrm{d}\rho_e}{\mathrm{d}z},$$

using volume conservation. Now add the two equations to obtain

$$w\frac{\mathrm{d}}{\mathrm{d}z}(4\pi/3)b^3(\rho - \rho_e) = -(4\pi/3)b^3 w\frac{\mathrm{d}\rho_e}{\mathrm{d}z}.$$

Hence

$$\frac{\mathrm{d}B}{\mathrm{d}z} = \frac{g}{\rho_0}\frac{\mathrm{d}\rho_e}{\mathrm{d}z}b^3 = -N^2 V,$$

which is the same as for the plume buoyancy flux.

Volume We now have a system for thermals and we note that the volume equation decouples from the others. We solve it and find

$$V = (V_0^{1/3} + \alpha_T z)^3,$$

where V_0 is the volume at $z = 0$. For a pure plume, with $V_0 = P_0 = 0$, we have $V = \alpha_T^3 z^3$. This is independent of stratification.

[1] The added mass can be calculated from the solution to potential flow around the object. For a sphere $M_a = M/2$; for a cylinder $M_a = M$. For other added masses, see e.g. *Marine hydrodynamics* (1977) by J. N. Newman. If the flow is very viscous or there are wave effects, things aren't to simple.

Buoyancy The buoyancy equation can also be solved explicitly. We find

$$B = B_0 - \int_0^z N^2(z')V(z')\,\mathrm{d}z'.$$

For an unstratified ambient, $B = B_0$ is constant.

Momentum This also has a closed solution, given by

$$P = \left[P_0^2 + (4/3)\int_0^z B(z')V(z')\,\mathrm{d}z'\right]^{1/2}.$$

Pure thermal The integrals can all be done exactly and we find that P is also a power law, given by

$$P = \frac{B_0^{1/2}}{\sqrt{3}}\alpha_T^{3/2}z^2.$$

Then the other quantities of the plume are found to be

$$b = \alpha_T z, \qquad w = \frac{B_0^{1/2}}{\sqrt{3}\alpha_T^{3/2}z}, \qquad g' = \frac{B_0}{\alpha_T^3 z^3}.$$

These results are consistent dimensionally. We see that on the whole these solutions are easier to obtain for the plume.

Comparison with experiment (Turner § 6.3.1) Scorer (1957) carried out laboratory experiments using thermals of heavy salt solution falling through fresh water, finding $b \sim t^{1/2}$ and $w \sim t^{-1/2}$. We need to re-express our solutions in terms of t. For the plume,

$$t = \int_0^z \frac{\mathrm{d}z'}{w} = \frac{\sqrt{3}}{2B_0^{1/2}}\alpha_T^{3/2}z^2,$$

i.e. $z \sim t^{1/2}$. Hence we have $b \sim t^{1/2}$ and $w \sim t^{-1/2}$, as found. However, this has to be true on dimensional grounds.

Power-law buoyancy frequency (Caulfield & Woods 1998) If $N^(z) = N_s^2(z/\lambda)^\beta$, where λ is some length scale, we find for a pure plume

$$B = B_0 - \frac{N_s^2\alpha_T^3 z^{4+\beta}}{\lambda^\beta(4+\beta)}.$$

We can hence find the height at which B vanishes if $4 + \beta > 0$. Next,

$$P = \frac{2}{\sqrt{3}}\left[\frac{1}{3}B_0\alpha_T^3 z^3 B_0 - \frac{N_s^2\alpha_T^6 z^{8+\beta}}{\lambda^\beta(4+\beta)(8+\beta)}\right]^{1/2}.$$

The specific momentum will vanish if $(4+\beta)(8+\beta) > 0$, although one should really rework the solution with $z_0 > 0$ to make sure that N^2 is well-behaved everywhere.