The differential equation is

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0.$$

The origin is a RSP, so write $y(x) = x^s \sum_{n=0}^{\infty} a_n x^n$ with $a_0 \neq 0$. The recurrence relation is then

$$[(n+s)^2 - \nu^2]a_n + a_{n-2} = 0 \qquad \text{for } n \geq 2.$$

The indicial and $O(x^{s-1})$ equations are

$$(s^2 - \nu^2)a_0 = 0 \qquad [(s+1)^2 - \nu^2]a_1 = 0.$$

So $s = \pm\nu$ and the recurrence relation becomes

$$n(n+2s)a_n + a_{n-2} = 0,$$

leading to the solution

$$J_{\pm\nu}(x) = \left(\frac{x}{2}\right)^{\pm\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n \pm \nu + 1)} \left(\frac{x}{2}\right)^{2n}.$$

where one adopts the conventional normalization

$$a_0 = \frac{1}{2^{\pm\nu}\Gamma(\pm\nu + 1)}.$$

The Gamma function is an extension of the factorial function with $\Gamma(n+s+1) = (n+s)(n+s-1)\dots(s+1)\Gamma(s+1)$.
The general theory of Frobenius series indicates that the second solution may not take Frobenius form if 2ν is an integer. In fact one sees that the series works when ν is a half-integer. For integer ν, it fails. The second series is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}[(s - s_1)y(s;x)]_{s=s_1},$$

where s_1 is the smaller of the indices and $y(s;x)$ denotes the Frobenius series in which the value of s is as yet unspecified. For the Bessel function, this means using the recurrence relation

$$a_n = -\frac{1}{(n+s+\nu)(n+s-\nu)}a_{n-2}.$$

For $\nu = 2$, this leads to

$$y(s;x) = x^s \left\{1 - \frac{x^2}{s(s+4)} + \frac{x^4}{s(s+4)(s+2)(s+6)} + \cdots\right\},$$

which is singular at $s = -2$. The second solution is given by

$$\begin{aligned} & \lim_{s\to -2} \frac{\mathrm{d}}{\mathrm{d}s} \left\{ (s+2)x^s \left[1 - \frac{x^2}{s(s+4)} + \cdots \right] \right\} \\ = \;& x^{-2} \log x \left[-\frac{x^4}{16} + \cdots \right] + x^{-2} \left[1 + \frac{x^2}{4} + \frac{x^4}{64} + \cdots \right] \\ = \;& -\frac{1}{4} J_2(x) \log x + \frac{1}{x^2} \left[1 + \frac{x^2}{4} + \cdots \right]. \end{aligned}$$

If $\nu = 0$, the indicial equation is $s^2 = 0$. Then

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \left(\frac{x}{2} \right)^{2n}.$$

The recurrence relation can be solved to give

$$a_{2n} = \frac{(-1)^n}{(2n+s)^2 (2n+s-2)^2 \cdots (2+s)^2} a_0.$$

The limiting procedure given above gives a second solution in the form

$$J_0(x) \log x - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \left(\frac{x}{2} \right)^{2n} \left[1 + \frac{1}{2} + \cdots + \frac{1}{n} \right].$$

In fact it is conventional to add another multiple of $J_0(x)$ to obtain a specific behavior as $x \to \infty$.