Final

This is a 3-hour open-note exam (no calculators, no books). Answer all six questions.

1 Analysis of Stokes flow past a sphere yields the ODE

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 f = 0.$$

Find the general solution. Now apply the boundary conditions $f(r) \sim -Ur^2/2$ for large r and $f(a) = f'(a) = 0$. [There is no paradox here.]

2 Consider the eigenvalue problem

$$f'' + \lambda f = 0.$$

on the interval $(0, 1)$ with $f(0) = 0$ and $f'(1) = f(1)$. Solve for the eigenfunctions. Obtain the equation

$$\tan \sqrt{\lambda_n} = \sqrt{\lambda_n}$$

for the eigenvalues λ_n. Show graphically that there are infinitely many positive eigenvalues. Write down the orthogonality relation between the different eigenfunctions $f_n(x)$.

3 Find the general series solution about $x = 0$ to the equation

$$w'' + 2xw' + 2w = 0.$$

Sum the series with $w(0) = 1$ and $w'(0) = 0$.

4 Find the general solution to the equation

$$y'' + \frac{1}{x}y' = \frac{1}{x^2y^3}.$$

[Hint: you could try a two-step reduction process, examining invariances at each step.]

5 Find the most general solutions $u(x, y)$ to the following equation, consistent with the boundary condition stated:

$$(x^2 + 1)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \qquad u(0, y) = \frac{1}{y^2 + 1}.$$

Where do you obtain a solution?

6 Solve Laplace's equation $\nabla^2 u = 0$ inside the rectangle with boundaries $x = 0$, $x = a$, $y = 0$, $y = b$, with boundary conditions $u = 1 - y/b$ on $x = 0$, $u = 1 - x/a$ on $y = 0$ and $u = 0$ everywhere else. [Hint: you can save time by solving the same problem twice.]