Final Solutions

1 This is an equidimensional equation. Taking $f = r^\alpha$ and applying the operator in parentheses once gives

$$[\alpha(\alpha-1)-2]r^{\alpha-2} = (\alpha+1)(\alpha-2)r^{\alpha-2}.$$

Applying the operator again leads to

$$(\alpha+1)(\alpha-2)[(\alpha-2)(\alpha-3)-2]r^{\alpha-4} = 0.$$

This factors to give

$$(\alpha-4)(\alpha-2)(\alpha-1)(\alpha+1) = 0.$$

The general solution is therefore

$$f = Ar^4 + Br^2 + Cr + \frac{D}{r}.$$

The boundary condition for large r forces $A = 0$ and $B = -U/2$. The solution now has the form $f = Cr + Dr^{-1} - (U/2)r^2$. Applying the condition at $r = a$ leads to $C = 3Ua/4$ and $D = -Ua^3/4$. The final result is

$$f = -\frac{U^2}{2}r^2 + \frac{3Ua}{4}r - \frac{Ua^3}{4r}.$$

2 The equation has constant coefficients and solution

$$f = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x.$$

The boundary condition at the origin requires $B = 0$, so the eigenfunctions are

$$f = A\sin\sqrt{\lambda}x.$$

The condition $f'(1) = f(1)$ leads to an equation for the eigenvalues λ_n:

$$A\sin\sqrt{\lambda_n} = A\sqrt{\lambda_n}\cos\sqrt{\lambda_n},$$

which can be written as

$$\tan\sqrt{\lambda_n} = \sqrt{\lambda_n}.$$

There are an infinite number of intersections of the curves $y = \tan x$ and $y = x$, then there are infinitely many solutions. The orthogonality relation between the different eigenfunctions is

$$\int_0^1 \sin(\sqrt{\lambda_m}x)\sin(\sqrt{\lambda_n}x)\,\mathrm{d}x = \frac{\delta_{mn}}{2}\sin^2\sqrt{\lambda_m}.$$

3 The origin is an ordinary point, so substitute in $w(x) = \sum_{n=0}^{\infty} a_n x^n$ to obtain

$$\sum_{n=0}^{\infty} [n(n-1)a_n x^{n-2} + 2na_n x^n + 2a_n x^n] = 0.$$

Shifting variables gives the recurrence relation

$$(n+2)(n+1)a_{n+2} + 2(n+1)a_n = 0,$$

i.e.

$$a_n = -\frac{2}{n} a_{n-2}.$$

From the recurrence relation, it is clear there is one even and one odd solution. The relation can be iterated to give

$$a_{2n} = (-1)^n \frac{2^n}{(2n)(2n-2)\cdots 2} a_0 = \frac{(-1)^n}{n!} a_0$$

for the even solution; this can be summed to give e^{-x^2}. The odd solution has coefficients

$$a_{2n+1} = (-1)^n \frac{2^n}{(2n+1)(2n-1)\cdots 3} a_1 = \frac{(-1)^n 2^n (2n)(2n-2)\cdots 2}{(2n+1)(2n)\cdots 2} a_1 = \frac{(-1)^n 2^{2n} n!}{(2n+1)!} a_1.$$

4 This is an equidimensional-in-x equation. Let $x = \mathrm{e}^t$ and find

$$y_{tt} - y_t + y_t + y^2 = y_{tt} + y^2 = \frac{1}{y^3}.$$

Now this is autonomous and we introduce $u = y_t$, which leads to

$$u \frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{y^3}.$$

Now separate variables and get

$$y_t = u(y) = \pm\sqrt{C - y^{-2}}.$$

Integrate and obtain

$$x(y) = \exp\left(\int^y \frac{\mathrm{d}\xi}{\pm[A - \xi^{-2})]^{1/2}} + B \right).$$

5 Characteristic equation:

$$\frac{\mathrm{d}x}{x^2+1} = \mathrm{d}y.$$

We find $p = y - \tan^{-1} x$ with $u = f(p)$. The boundary condition gives $f(y) = 1/(1+y^2)$, so

$$u(x,y) = \frac{1}{(y - \tan^{-1} x)^2 + 1}.$$

The characteristic curves

$$x = \tan(y - p),$$

cover the whole plane.

6 Solve Laplace's equation with boundary conditions $u = 1 - y/b$ on $x = 0$ and $u = 1 - x/a$ on $y = 0$. The solution for the first case

$$u_1 = \sum_{m=1}^{\infty} A_m \sin \frac{m\pi y}{b} \sinh \frac{m\pi(x-a)}{b}.$$

Computing the coefficients in the Fourier series gives

$$A_m = -\frac{2}{b \sinh(m\pi a/b)} \int_0^b (1 - \frac{y}{b}) \sin \frac{m\pi y}{b} \, \mathrm{d}y = \frac{2}{m\pi \sinh(m\pi a/b)}.$$

The solution to the second problem can be written down immediately as

$$u_2 = \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{a} \sinh \frac{m\pi(y-b)}{a}$$

with

$$B_m = -\frac{2}{b \sinh(m\pi b/a)} \int_0^b (1 - \frac{x}{a}) \sin \frac{m\pi x}{a} \, \mathrm{d}x = \frac{2}{m\pi \sinh(m\pi b/a)}.$$

The total solution is

$$u(x,y) = u_1 + u_2.$$