Homework III

Due Oct 26, 2016.

1 Which of the following series solutions of the equation

$$y'' + \frac{\sin 3x}{\sin 2x^2} y' + \frac{\cos x - 1}{x^4} y = 0$$

are incorrect and which might be correct?

$$\begin{aligned} y_1(x) &= x^{-1} + \cdots + \mathrm{e}^{-1/x^2}[1 + \cdots], \\ y_2(x) &= x^{-1} + \cdots + x^{-1} \log x [1 + \cdots], \\ y_3(x) &= x^{1/2} + \cdots, \\ y_4(x) &= 1 + \cdots. \end{aligned}$$

Your answer should only require one or two lines of algebra.

2 Find two series solutions about $x = 0$ to the equation

$$y'' - \frac{2}{2x+1} y' - \frac{2x+3}{2x+1} y = 0.$$

Sum them.

3 Chebyshev's equation is

$$(1 - x^2) y'' - x y' + p^2 y = 0.$$

Obtain series solutions about $x = 0$ (you may leave the answer in terms of a recurrence relation) and show that one of therm terminates when p is an integer. Make the substitution $x = \cos\theta$ and solve the equation exactly in terms of θ. Verify your series solutions for $p = 0, 1, 2$ and 3.

4 Solve the equation

$$y' - \frac{1 - x^2}{x} y = 0, \qquad y'(0) = 1$$

as a series expansion about the origin. Solve the problem exactly using an integrating factor and show that the answers match.

5 Discuss the nature of the point $x = 1$ in Legendre's equation

$$[(1 - x^2)f']' + l(l + 1)f = 0.$$

Find the recurrence relation for the series solution about $x = 1$ with the larger index (use the variable $y = x - 1$). Show that these solutions terminate if l is an integer and obtain them for $l = 0, 1$ and 2.

6 Show that the origin is a regular singular point for the differential equation

$$y'' + \frac{1}{x}y = 0.$$

Solve the indicial equation and show that the solution associated with the larger index is

$$y_1 = x \sum_{n=0}^{\infty} \frac{(-x)^n}{(n + 1)(n!)^2}.$$

Make the change of variable $y = \sqrt{x}f(2\sqrt{x})$ and solve the resulting equation for f. What does this tell you about the nature of the second series solution? [Note: Bessel's equation in the form

$$f'' + \frac{1}{r}f' + \left(1 - \frac{n^2}{r^2}\right) f = 0$$

has solutions $J_n(r)$ and $Y_n(r)$. Near the origin, $J_1 = r/2 + \cdots$ and $Y_1(r) = -(\pi r/2)^{-1} + r(\log r + A) + \cdots$.]