Midterm Solution

1 The origin is an RSP with indicial equation

$$s(s-1)+\frac{3}{2}s-\frac{1}{2}=s^2+\frac{1}{2}s-\frac{1}{2}.$$

This has roots -1 and $1/2$. Of the three curves, only y_1 is consistent with this behavior with a vertical slope at the origin corresponding to $x^{1/2}$.

2 Try $y=x^a$. This gives

$$a(a-1)x^{a-2}+\frac{ax^a}{3(x+3)}-\frac{x^a}{3(x+3)}=0,$$

which is satisfied by $a=1$. For the other solution, write $y=xu$. Then

$$xu''+2u'+\frac{x}{3(x+3)}(xu'+u)-\frac{1}{3(x+3)}xu.$$

Simplifying and dividing by x gives

$$u''+u'\left(\frac{2}{x}+\frac{1}{3}-\frac{1}{x+3}\right)=0.$$

The integrating factor is $x^2(x+3)^{-1}e^{x/3}$. Hence $u=(x^{=1}+3x^{-2})e^{-x/3}$. This integrates to give $u=-3x^{-1}e^{-x/3}$. So we take as second solution $e^{-x/3}$. A simpler way is to substitute $y=e^{rx}$ into the governing equation and obtain

$$\left(r^2+\frac{rx}{3(x+3)}-\frac{1}{3(x+3)}\right)e^{rx}=\frac{3(x+3)r^2+rx-1}{3(x+3)}e^{rx}=0,$$

which is satisfied by $r=-1/3$. Now construct the Green's function, which takes the form

$$G(x;a)=A\begin{cases}xe^{-a/3} & x<a,\\ e^{-x/3}a & x>a.\end{cases}$$

Then the jump condition becomes $A[-(e^{-x/3}/3)a-e^{-a/3}]_{x=a}=-Ae^{-a/3}(a+3)/3=1$. This gives $A=-3e^{a/3}/(a+3)$, so that

$$y(x)=-3\int_0^x\frac{a}{a+3}e^{(a-x)/3}f(a)\,da-3\int_x^\infty\frac{x}{a+3}f(a)\,da.$$

3 In Sturm–Liouville form the equation becomes

$$-(x^2y')' = \lambda x^2 y,$$

so $p = x^2$, $q = 0$ and $w = x^2$. Leibniz's rule gives $(xy)'' = xy'' + 2y'$, which leads to the result shown. Hence the differential equation becomes

$$(xy)'' + \lambda(xy) = 0.$$

Considered as an equation for $u = xy$, this has solutions

$$u = xy = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x.$$

The boundary condition at the origin requires $B = 0$, so the eigenfunctions are

$$y = A\frac{\sin\sqrt{\lambda}x}{x}.$$

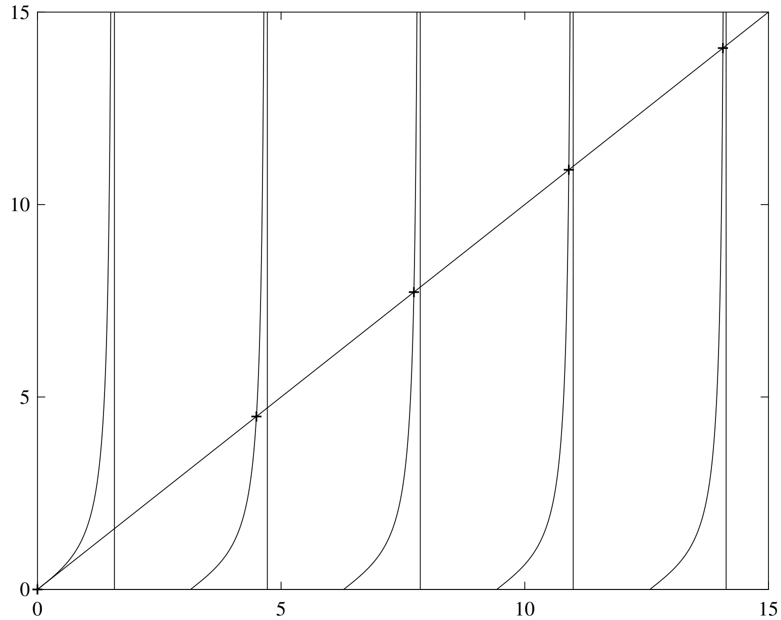
For the boundary condition, calculate

$$y' = A\left(\lambda\frac{\cos\sqrt{\lambda}x}{x} - \frac{\sin\sqrt{\lambda}x}{x^2}\right).$$

The condition $y'(1) = 0$ leads to an equation for the eigenvalues λ_n:

$$\tan\sqrt{\lambda_n} = \sqrt{\lambda_n}$$

Figure 1 shows the first several solutions for $\sqrt{\lambda_n}$ as intersections of $\tan x$ and x. There are infinitely many solutions; for large n they look like $\lambda_n \sim (n+1/2)^2\pi^2$. The eigenvalue relation is satisfied by positive λ_n, since negative λ_n turn the tan into tanh and $\tanh x < x$ for $x > 1$. However, the function $y = A$ (constant) is an eigenfunction with eigenvalue 0.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