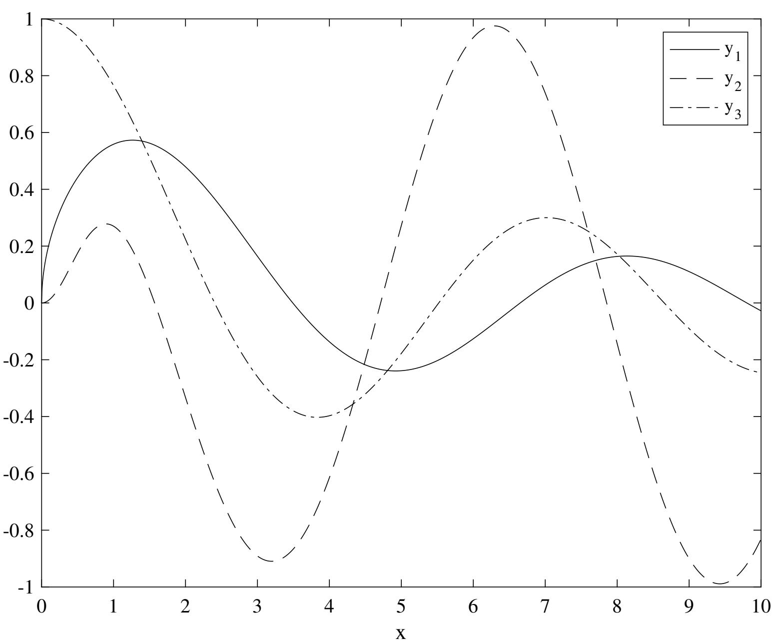
Midterm

This is a 50 minute open-note exam (no calculators, no books). Answer all three questions and justify your answers.

1 (5 points) Professor Ll has solved the equation

$$y'' + \frac{3}{2x}y' + \left(1 - \frac{1}{2x^2}\right)y = 0$$

on the interval $(0, 10)$, but has forgotten which of the following curves it is. Which curve is it? Explain your choice. You should only need to write about one sentence and one line of algebra to do this.



2 (10 points) Find two homogeneous solutions of the equation

$$y'' + \frac{x}{3(x+3)}y' - \frac{1}{3(x+3)}y = 0.$$

[Hint: look for a simple solution, e.g. a polynomial, and then use reduction in order.] Hence construct the Green's function for the boundary value problem

$$y'' + \frac{x}{3(x+3)}y' - \frac{1}{3(x+3)}y = f(x), \qquad y(0) = 0, \quad y(\infty) = 0,$$

and show that the solution to the problem is

$$y(x) = -3\int_0^x \frac{a}{a+3}\mathrm{e}^{(a-x)/3} f(a)\,\mathrm{d}a - 3\int_x^\infty \frac{x}{a+3} f(a)\,\mathrm{d}a.$$

3 (10 points) Consider the eigenvalue problem

$$y'' + \frac{2}{x}y' + \lambda y = 0.$$

on the interval $(0, 1)$ with $y(0)$ bounded and $y'(1) = 0$. Write the equation in Sturm–Liouville form, identifying p, q and w. Show that

$$y'' + \frac{2}{x}y' = \frac{1}{x}(xy)'',$$

and hence or otherwise solve for the eigenfunctions. Obtain the equation

$$\tan\sqrt{\lambda_n} = \sqrt{\lambda_n}$$

for the eigenvalues λ_n. Show graphically that there are infinitely many positive eigenvalues. Is $\lambda = 0$ an eigenvalue? If so, what is the corresponding eigenfunction?