Solutions 0

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$$(a)\quad z_1 = \frac{\cos 4\theta - 4 - \mathrm{i}\sin 4\theta}{5 - 4\cos 2\theta}, \qquad (b)\quad z_2 = -\frac{3+\mathrm{i}}{2}, \qquad (c)\quad z_3 = (\mathrm{e}^{\mathrm{i}\pi/3})^{2016} = 1.$$

2 *(a)* A particular solution is $y = \frac{1}{2}\mathrm{e}^x$. The homogeneous solution is $A\sin x + B\cos x$. Hence the full solution is

$$y = A\sin x + B\cos x + \frac{1}{2}\mathrm{e}^x.$$

Plugging in the boundary conditions,

$$y = -\frac{1}{2}\sin x - \frac{1}{2}\cos x + \frac{1}{2}\mathrm{e}^x.$$

(b) We find the integrating factor $\mathrm{e}^{-x^3/3}$, so after some algebra and applying the boundary condition,

$$y = \mathrm{e}^{x^3/3}\int_0^x \mathrm{e}^{-t^3/3}\sin t\,\mathrm{d}t.$$

(c) We have $yy'' + y'^2 = (yy')'$, so $yy' = 2x + A$. Integrate again, giving $\frac{1}{2}y^2 = x^2 + Ax + B$. The boundary conditions lead to $A = B = 0$, so the solution appears to be $y = \pm\sqrt{2}x$. However, these boundary conditions are inconsistent with the original differential equation, so there is no solution. It is worth investigating which boundary conditions are acceptable.
(d) We separate variables with $(1+y)\,\mathrm{d}y = \mathrm{d}x$. Hence $y + \frac{1}{2}y^2 = x + A$. Applying the boundary condition gives $A = 0$. Hence $y + \frac{1}{2}y^2 = x$, which can be solved to give $y = -1 \pm \sqrt{1+2x}$. We need the plus sign to satisfy the boundary condition, so $y = -1 + \sqrt{1+2x}$

3 Solving $\det(A - \lambda I) = 0$ gives a repeated root $\lambda_1 = \lambda_2 = 1$. There is only one eigenvector: $[0,1]^T$.

4 The function x^{-1} has a singularity at $x = 0$, but $\tanh x = x - x^3/3 + \ldots$ for small x, so $\lim_{x\to 0}(x^{-1}\tanh x) = 1$. See Figure 1.

5 Use Euler's formula:

$$\sin 4x = \mathrm{Im}\,(\mathrm{e}^{4\mathrm{i}x}) = \mathrm{Im}\,[(\cos x + \mathrm{i}\sin x)^4] = 4\sin x\cos^3 x - 4\sin^3 x\cos x.$$

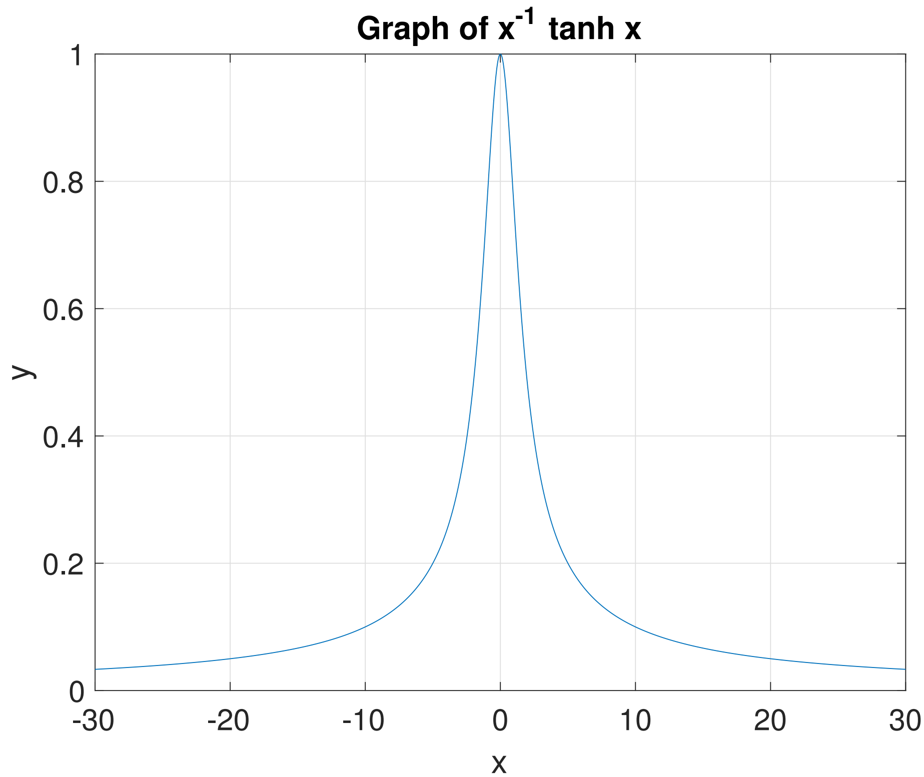

Figure 1: Problem 4.

6 Substitute in $\lambda g(\lambda x)$ into the equation. This gives

$$\lambda^4 g''' + (\lambda g)(\lambda^3 g'') = \lambda^4(g''' + gg'') = 0$$

so the equation is satisfied. Solve the IVP with $f(0) = f'(0) = 0$, $f''(0) = 1$. For large x, this has $f' \to \mu$. Now $f' = \lambda^2 g' = \lambda^2$, where g is the solution with $g' \to 1$ for large x. Hence $\lambda = \mu^{-1/2}$ and $g''(0) = \lambda^3$.

The following Matlab code carries out this procedure (the warning is harmless):

```
format long
options = odeset('RelTol',eps,'AbsTol',eps);
f0 = @(t,y) [y(2) ; y(3) ; -y(1)*y(3)];
[x,f] = ode45(f0,[0 10],[0 0 1],options);
l = sqrt(1/f(end,2)); g0pp = l^3
```

Running this gives $f''(0) = 0.469599988361012$.