Solutions I

1 Constant coefficients, so seek solutions of the form e^{rx}. The associated polynomial is $r^3 + 1 = 0$, so $r = e^{i\pi/3}, e^{i\pi} = -1, e^{-i\pi/3}$. The general solution is then

$$y = A e^{x e^{i\pi/3}} + B e^{-x} + C e^{x e^{5i\pi/3}}.$$

This can be rewritten as

$$\begin{aligned} y &= A e^{x/2 + ix\sqrt{3}/2} + B e^{-x} + C e^{x/2 - ix\sqrt{3}/2} \\ &= \alpha e^{x/2} \cos(x\sqrt{3}/2) + \beta e^{x/2} \sin(x\sqrt{3}/2) + B e^{-x}, \end{aligned}$$

where $\alpha = A + C$ and $\beta = A - C$.

2 Constant coefficients again. The polynomial is $r^2 + 4 = 0$, so $r = \pm 2i$ and the general solution can be written as $y = A e^{2ix} + B e^{-2ix}$ or $y = C \cos 2x + D \sin 2x$. Applying the boundary condition at $x = 0$, we find that $C = 0$. Applying the condition at $x = 1$, we find that $D = (\sin 2)^{-1}$. The solution is

$$y = \frac{\sin 2x}{\sin 2}.$$

3 Equidimensional equation, so try a solution $y = x^\alpha$. The polynomial gives $\alpha^3 + 1 = 0$. This is the same as in **1** with $\alpha = e^{i\pi/3}, e^{i\pi} = -1, e^{-i\pi/3}$. This leads to

$$y = A x^{e^{i\pi/3}} + \frac{B}{x} + C x^{e^{5i\pi/3}} = \alpha x^{1/2} \cos(\sqrt{3}/2 \log x) + \beta x^{1/2} \sin(\sqrt{3}/2 \log x) + B x^{-1}.$$

4 By inspection $y = 1$ is a particular integral. The homogeneous equation is equidimensional. Substituting $y = x^\alpha$ gives $\alpha^2 + 4\alpha + 5 = (\alpha + 2)^2 + 1 = 0$. Hence $\alpha = -2 \pm i$ and the solution is $y = A x^{-2+i} + B x^{-2-i} + 1$. The boundary conditions give

$$A + B + 1 = 1, \qquad (-2 + i)A + (-2 - i)B = 1.$$

This gives $B = -A$ and $2iA = 1$. Hence

$$y(x) = \frac{x^{-2+i}}{2i} - \frac{x^{-2-i}}{2i} + 1 = \frac{\sin(\log x)}{x^2} + 1.$$

5 First-order equation, so use the integrating factor $e^{\int^x e^{-x'}\,dx'}$:

$$\left(e^{-e^{-x}} y\right)' = e^{-e^{-x}} \sin x.$$

Integrating gives

$$y(x) = e^{e^{-x}} \left(A + \int^x e^{-e^{-x'}} \sin x' \, dx' \right).$$

6 Applying the operator to f gives the equidimensional equation

$$f^{(4)} + \frac{2f'''}{r} - \frac{3f''}{r^2} + \frac{3f'}{r^3} - \frac{3f}{r^4} = 0.$$

For the solution $f = r^\alpha$, the resulting polynomial factors to give

$$(\alpha - 3)(\alpha - 1)^2(\alpha + 1) = 0.$$

A simpler way to arrive at this equation for α is to notice that the operator is going to result in an equidimensional equation. Taking $f = r^\alpha$ and applying the operator once gives

$$(\alpha(\alpha - 1) + \alpha - 1)r^{\alpha-2} = (\alpha - 1)(\alpha + 1)r^{\alpha-2}.$$

Applying the operator again leads to

$$(\alpha - 1)(\alpha + 1)\left[(\alpha - 2)(\alpha - 3) + (\alpha - 2) - 1\right] r^{\alpha-4} = 0.$$

This clearly factors to the same thing found above:

$$(\alpha - 3)(\alpha - 1)^2(\alpha + 1) = 0.$$

The general solution is therefore

$$f = Ar^3 + Br \ln r + Cr + \frac{D}{r}.$$

The boundary condition for large r forces $A = B = 0$. Our solution now has the form $f = Cr + D/r$. Applying the condition on $f(a)$, we find that $D = -Ca^2$. However, applying the condition in $f'(a)$ leads to $D = Ca^2$. The only way these can both be true is if $C = D = 0$. This means there is no non-trivial solution to the problem.