Solutions II

1 The equation is

$$y'' + p_1(x)y' + p_0(x) = 0,$$

where $p_1(x) = \log x$, $p_0(x) = -1 - \log x$. Since $\sum p_k = 0$, $u = \mathrm{e}^x$ is a solution. Then

$$y = uv, y' = u'v + uv', \qquad y'' = u''v + 2u'v' + uv''.$$

So $u''v + 2u'v' + uv'' + p_1(u'v + uv') + p_0uv = v(u'' + p_1u' + p_0u) + (uv'' + 2u'v' + p_1uv') = 0$. This gives $uv'' + 2u'v' + p_1uv' = 0$ which is a first-order ODE for v'. Separate variables:

$$\begin{aligned} \frac{v''}{v'} &= -\frac{2u' + p_1u}{u} = -\frac{2u'}{u} - p_1u, \\ \log v' &= -2\log u - \int_{x_0}^{x} p_1(x')\,\mathrm{d}x' + C_1, \\ v' &= C_2\frac{\mathrm{e}^{-\int_{x_0}^{x} p_1(x')\,\mathrm{d}x'}}{u^2} = C_2\frac{\mathrm{e}^{-\int_{x_0}^{x} \log x'\,\mathrm{d}x'}}{\mathrm{e}^{2x}} = C_3\mathrm{e}^{-x-x\log x}. \end{aligned}$$

Thus

$$v = C_3\int_{x_0}^{x} \mathrm{e}^{-x'-x'\log x'}\,\mathrm{d}x' + C_4$$

and

$$y = C_3\mathrm{e}^x\int_{x_0}^{x} \mathrm{e}^{-x'-x'\log x'}\,\mathrm{d}x' + C_4\mathrm{e}^x.$$

2 Looking for simple polynomials shows that $y = x$ is a solution. Writing $y = xu$ gives

$$u'' + \left(\frac{2}{x} + xp\right)u' = 0.$$

This has integrating factor

$$\exp\int^x [2b^{-1} + bp(b)]\,\mathrm{d}b = x^2\exp\int^x bp(b)\,\mathrm{d}b.$$

Hence a second solution is

$$y(x) = x\int^x a^{-2}\exp\left(-\int^a bp(b)\,\mathrm{d}b\right)\mathrm{d}a.$$

The lower limits are arbitrary. For $p(x) = \cos x$, the integral is $-\int^x a \cos a \, \mathrm{d}a = -x \sin x - \cos x$. Then

$$y(x) = Ax \int_\pi^x a^{-2} \mathrm{e}^{-a \sin a - \cos a} \, \mathrm{d}a.$$

is a solution satisfying the boundary condition at $x = \pi$. Applying the second boundary condition gives

$$y(x) = \frac{x \displaystyle\int_\pi^x a^{-2} \mathrm{e}^{-a \sin a - \cos a} \, \mathrm{d}a}{2\pi \displaystyle\int_\pi^{2\pi} a^{-2} \mathrm{e}^{-a \sin a - \cos a} \, \mathrm{d}a};$$

note that the denominator is just a number.

3 The Green's function for this problem satisfies

$$G' + G \sin x = \delta(x - a).$$

The homogeneous solution is $\exp(\cos x)$, which comes e.g. from an integrating factor. The Green's function is

$$G(x; a) = \begin{cases} A \exp \cos x & x < a, \\ B \exp \cos x & x > a. \end{cases}$$

Consider $x \geq 0$. From the boundary condition at $x = 0$, $A = 0$. The jump condition gives $B \exp(\cos a) = 1$, so

$$G(x; a) = \begin{cases} 0 & x < a, \\ \exp(\cos x - \cos a) & x > a. \end{cases}$$

Our final solution is therefore

$$y(x) = \int_0^x f(a) \mathrm{e}^{\cos x - \cos a} \, \mathrm{d}a.$$

Using the integrating factor $\exp(-\cos x)$ directly gives

$$y = A\mathrm{e}^{\cos x} + \int_0^x f(a) \mathrm{e}^{\cos x - \cos a} \, \mathrm{d}a.$$

Applying the BC, $y(0) = A\mathrm{e} = 0$, so $A = 0$ and we recover the same solution as above.

4 The homogenous equation has constant coefficients, so with $y = \mathrm{e}^{rx}$,

$$r^2 + 2r + 1 = (r + 1)^2 = 0,$$

and hence

$$y_h = C_1 \mathrm{e}^{-x} + C_2 x \mathrm{e}^{-x}.$$

The Green's function satisfies

$$\frac{\mathrm{d}G^2(x, z)}{\mathrm{d}x^2} + 2\frac{\mathrm{d}G(x, z)}{\mathrm{d}x} + G(x, z) = \delta(x - z),$$

with solution

$$G(x;z) = \begin{cases} A(z)\mathrm{e}^{-x} + B(z)x\mathrm{e}^{-x} & x < z, \\ C(z)\mathrm{e}^{-x} + D(z)x\mathrm{e}^{-x} & x > z. \end{cases}$$

Apply BCs: $y(0) = 0$, so $A(z) = 0$; and then $\lim_{x\to\infty} \mathrm{e}^x y(x)$ is bounded, so $D(z) = 0$. Hence

$$C(z)\mathrm{e}^{-z} - B(z)\cdot z\cdot \mathrm{e}^{-z} = 0, \qquad -C(z)\mathrm{e}^{-z} - \left[B(z)\mathrm{e}^{-z} - B(z)\cdot z\cdot \mathrm{e}^{-z}\right] = 1.$$

This leads to

$$B(z) = -\mathrm{e}^z, \qquad C(z) = -z\mathrm{e}^z.$$

Finally we obtain

$$G(x;z) = \begin{cases} -\mathrm{e}^z x\mathrm{e}^{-x} & x < z, \\ -z\mathrm{e}^z\mathrm{e}^{-x} & x > z. \end{cases}$$

and

$$\begin{aligned} y(x) &= -\mathrm{e}^{-x}\int_0^x \frac{z\mathrm{e}^z}{1+\mathrm{e}^{2z}}\mathrm{d}z - x\mathrm{e}^{-x}\int_x^\infty \frac{\mathrm{e}^z}{1+\mathrm{e}^{2z}}\mathrm{d}z \\ &= -\mathrm{e}^{-x}\int_0^x \frac{z\mathrm{e}^z}{1+\mathrm{e}^{2z}}\mathrm{d}z - x\mathrm{e}^{-x}\left(\frac{\pi}{2} - \tan^{-1}\mathrm{e}^x\right). \end{aligned}$$

5 Two independent solutions are r and r^{-1}, with Wronskian $-2r^{-1}$. The Green's function is

$$G(x;s) = \begin{cases} A(s)r + B(s)/r & x < s, \\ C(s)r + D(s)/r & x > s. \end{cases}$$

Apply the BC: $f(0) = 0$, so $B(s) = 0$. Hence the jump conditions give

$$C(s)s + D(s)/s - A(s)s = 0, \qquad C(s) - D(s)/s^2 - A(s) = 1.$$

This leads to

$$C(s) = A(s) + 1/2, \qquad D(s) = -\frac{s^2}{2}.$$

Finally we obtain

$$G(x;s) = \begin{cases} A(s)r & x < s, \\ (A(s)+1/2)r - \frac{s^2}{2r} & x > s. \end{cases}$$

and

$$\begin{aligned} f(r) &= \int_0^r \left[(A(s)+1/2)r - \frac{s^2}{2r}\right] g(s)\,\mathrm{d}s + r\int_r^R A(s)g(s)\,\mathrm{d}s \\ &= r\int_0^R A(s)g(s)\,\mathrm{d}s + \frac{r}{2}\int_0^r g(s)\,\mathrm{d}s - \frac{1}{2r}\int_0^r s^2 g(s)\,\mathrm{d}s \\ &= A'r + \frac{r}{2}\int_0^r g(s)\mathrm{d}s - \frac{1}{2r}\int_0^r s^2 g(s)\mathrm{d}s. \end{aligned}$$

The constant R would correspond to the location of the second boundary. We see that we obtain an arbitrary multiple of the homogeneous solution r; the constant A' would come from applying a second bounadry condition. We need $g(s)$ to look like s^a with $a > -1$ near the origin for the $g(s)$ integral to exist there. If R is finite, the last integral is well-behaved at that right endpoint. If R is infinite, we need $g(s) \sim s^a$ with $a < -3$ for large s for the last integral to exist. (Logarithmic corrections do not affect convergence.)

6 Consider $x < 0$ and $x > 0$ separately. For $x < 0$ we see

$$y'' - y = \mathrm{e}^x.$$

The RHS is a homogeneous solution, so try $\alpha x \mathrm{e}^x$. We find $2\alpha \mathrm{e}^x = \mathrm{e}^x$, so $\alpha = 1/2$. The other homogeneous solution is e^{-x} so we have a general solution

$$y = A\mathrm{e}^x + B\mathrm{e}^{-x} + \frac{x}{2}\mathrm{e}^x.$$

For $x > 0$ we see

$$y'' + y = \mathrm{e}^{-x}.$$

Now if we try $\beta \mathrm{e}^{-x}$ as a solution we get $2\beta \mathrm{e}^{-x} = \mathrm{e}^{-x}$ so $\beta = 1/2$. The homogeneous solutions are $\sin x$ and $\cos x$ so we have a general solution

$$y = \frac{\mathrm{e}^{-x}}{2} + C \sin x + D \cos x.$$

The equation is second order, so the first and second derivatives must be continuous at $x = 0$ in order for the second derivative to satisfy the differential equation. Matching the solutions yields

$$B = A - C + 1, \qquad D = 2A - C + \frac{1}{2},$$

and our general solution is

$$\begin{aligned} y(x<0) &= A\mathrm{e}^x + (A - C + 1)\mathrm{e}^{-x} + \frac{x}{2}\mathrm{e}^x \\ y(x>0) &= C \sin x + \left(2A - C + \frac{1}{2}\right) \cos x + \frac{\mathrm{e}^{-x}}{2}. \end{aligned}$$