Solutions III

1 Investigating the behavior near the origin shows that $p_1 \sim (3/2)x^{-1}$ and $p_0 \sim -(1/2)x^{-2}$. Hence the origin is a regular singular point and (a) is not possible since it is not a Frobenius series. The indicial equation is $s(s-1) + 3s/2 - 1/2$, which has roots -1 and $1/2$. Hence (b) is impossible since the roots do not differ by an integer and (d) is impossible since 0 is not a root. The only possibility is (c).

2 The origin is an ordinary point. Multiply through by $2x+1$ and substitute in $y(x) = \sum_{n=0}^{\infty} a_n x^n$:

$$\sum_{n=0}^{\infty} [n(n-1)a_n(2x^{n-1} + x^{n-2}) - 2na_n x^{n-1} - a_n(2x^{n+1} + 3x^n)] = 0.$$

Shifting variables gives the recurrence relation

$$\begin{aligned} & 2(n+1)na_{n+1} + (n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} - 2a_{n-1} - 3a_n \\ = \ & (n+2)(n+1)a_{n+2} + 2(n+1)(n-1)a_{n+1} - 3a_n - 2a_{n-1} = 0, \end{aligned}$$

where a_{-1} is taken to be zero in the $n = 0$ equation. There are two solutions. One can be obtained from $a_0 = 0$ and $a_1 = 1$; this leads to the series

$$x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \cdots$$

which looks like $x\mathrm{e}^x$. If one then substitutes $a_n = 1/(n-1)!$ for $n \geq 1$, one obtains

$$\frac{n+2}{n!} + \frac{2(n+1)(n-1)}{n!} - \frac{3}{(n-1)!} - \frac{2}{(n-2)!} = \frac{1}{n!}\left[n + 2 + 2(n^2-1) - 3n - 2(n^2-n)\right]$$

for $n \geq 1$, which vanishes, and $2a_2 - 2a_1 - 3a_0 = 2 - 2 = 0$. Hence one solution is indeed $x\mathrm{e}^x$. For the second, take $a_0 = 1$ and $a_1 = -1$; this leads to the series

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

which looks like e^{-x}. If one then substitutes $a_n = (-1)^n/n!$, one obtains

$$\frac{(-1)^{n+2}}{n!} + \frac{2(n-1)(-1)^{n+1}}{n!} - \frac{3(-1)^n}{n!} - \frac{2(-1)^{n-1}}{(n-1)!} = \frac{(-1)^n}{n!}[1 - 2(n-1) - 3 + 2n]$$

for $n \geq 1$, which vanishes, and $2a_2 - 2a_1 - 3a_0 = 1 + 2 - 3 = 0$. Hence another solution is indeed e^{-x}.

3 The origin is an ordinary point, so substitute in $y(x) = \sum_{n=0}^{\infty} a_n x^n$, yielding the recurrence relation

$$a_{n+2} = \frac{n^2 - p^2}{(n+2)(n+1)} a_n.$$

Hence if p is an integer, one series terminates (the even series if p is even and the odd series if p is odd). The series for p up to 3 are

$$y_0 = 1, \qquad y_1 = x, \qquad y_2 = 1 - 2x^2, \qquad y_3 = x - 4x^3/3.$$

Starting from $x = \cos\theta$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{x_\theta}\frac{\mathrm{d}}{\mathrm{d}\theta} = -\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2} = -\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\right) = \frac{1}{\sin^2\theta}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} - \frac{\cos\theta}{\sin^3\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}.$$

The equation transforms to

$$(1 - \cos^2\theta)\left(\frac{1}{\sin^2\theta} y_{\theta\theta} - \frac{\cos\theta}{\sin^3\theta} y_\theta\right) + \frac{\cos\theta}{\sin\theta} y_\theta + p^2 y = y_{\theta\theta} + p^2 y = 0,$$

the simple harmonic oscillator equation. This has solution $y = A\cos p\theta = \cos\left(p\cos^{-1}x\right)$ and $y = B\sin p\theta = B\sin\left(p\cos^{-1}x\right)$. The former solution leads to polynomials if p is an integer. Taking $A = 1$ gives

$$y_0 = 1, \qquad y_1 = \cos\theta = x, \qquad y_2 = \cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1,$$

and

$$\begin{aligned} y_3 &= \cos 3\theta = \cos\theta\cos 2\theta - \sin\theta\sin 2\theta = \cos\theta\cos 2\theta - 2\sin^2\theta\cos\theta \\ &= x(2x^2 - 1) - 2(1 - x^2)x = 4x^3 - 3x, \end{aligned}$$

clearly the same as before up to normalization

4 The origin is an RSP. Substituting in the Frobenius form gives

$$\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n[x^{n+s-1} - x^{n+s+1}] = 0.$$

Hence the indicial equation is $s - 1 = 0$ and hence $s = 1$. This gives successively $a_1 = 0$ and the recurrence relation

$$(n+2)a_{n+2} + a_n = 0.$$

This can be solved to give

$$a_n = -\frac{1}{n}a_{n-2} = \cdots = \frac{(-1)^{n/2}}{n(n-2)\cdots 4\cdot 2}a_0 = \frac{(-1)^{n/2}}{2^{n/2}(n/2)(n/2-1)\cdots 2\cdot 1}a_0 = \frac{(-1)^{n/2}}{2^{n/2}(n/2)!}a_0,$$

since n is even. Hence the series becomes

$$y(x) = x a_0 \sum_{n=0}^{\infty}\frac{(-x^2/2)^m}{m!} = x\mathrm{e}^{-x^2/2}a_0.$$

The integrating factor is $\exp\int(-x^{-1} + x)\,\mathrm{d}x = \exp\left(-\log x + x^2/2\right) = x^{-1}\mathrm{e}^{x^2/2}$, so the solution is $y(x) = Ax\mathrm{e}^{-x^2/2}$, as above.

5 The point $x = 1$ is an RSP, as is $x = -1$. Substitute the Frobenius series $(x-1)^s \sum_{n=0}^{\infty}(x-1)^n$ into the equation. Write $y = x - 1$, so that $1 - x^2 = -2y - y^2$, so the recurrence relation becomes

$$-2\sum_{n=0}^{\infty}(n+s)^2 a_n y^{n+s-1} - \sum_{n=0}^{\infty}[(n+s)(n+s+1) - l(l+1)]a_n y^{n+s} = 0.$$

The indicial equation is $s^2 = 0$, so there is one Frobenius (actually Taylor) series with $s = 0$. Solving the recurrence relation gives

$$a_{n+1} = \frac{l(l+1) - n(n+1)}{2(n+1)^2}a_n.$$

This terminates when l is an integer. The first three solutions are

$$1 = P_0(x), \qquad y + 1 = x = P_1(x), \qquad \frac{3}{2}y^2 - 3y + 1 = \frac{3}{2}x^2 - \frac{1}{2} = P_2(x),$$

as before. Since the indicial root has two identical solutions, the second solution must be logarithmic.

6 We have $p_1(x) = 0$ and $x^2 p_0(x) = x$, both of which are finite as $x \to 0$, so the origin is an RSP. Substituting in the Frobenius form gives

$$\sum_{n=0}^{\infty}(n+s)(n+s-2)a_n x^{n+s-2} + \sum_{n=0}^{\infty} a_n x^{n+s-1} = 0.$$

Taking $n = 0$ gives the indicial equation $s(s-1) = 0$ and hence $s = 0, 1$. The larger of these will give a Frobenius solution; the smaller might have a logarithmic term since the solutions to the indicial equation differ by an integer. Taking $s = 1$ gives the recurrence relation

$$(n+2)(n+1)a_{n+1} + a_n = 0.$$

This leads to

$$a_n = -\frac{a_{n-1}}{(n+1)n} = \frac{a_{n-2}}{(n+1)n^2(n-1)} = \cdots = \frac{(-1)^n}{(n+1)n^2(n-1)^2\cdots 2}a_0 = \frac{(-1)^n}{(n+1)(n!)^2}a_0.$$

With the given change of variable, the chain rule gives

$$y' = \frac{1}{2\sqrt{x}}f + f', \qquad y'' = -\frac{1}{4x^{3/2}}f + \frac{1}{2x}f' + \frac{1}{\sqrt{x}}f''.$$

Hence, with $r = 2\sqrt{x}$,

$$\frac{1}{\sqrt{x}}f'' + \frac{1}{2x}f' + \left(\frac{1}{\sqrt{x}} - 1\frac{1}{4x^{3/2}}\right)f = \frac{1}{\sqrt{x}}\left[f'' + \frac{1}{r}f' + \left(1 - \frac{1}{r^2}\right)f\right] = 0.$$

This is Bessel's equation with $n = 1$, so the solution can be written as

$$y(x) = A\sqrt{x}J_1(\sqrt{x}) + B\sqrt{x}Y_1(2\sqrt{x}).$$

From the expansion of $Y_1(r)$, the second series in the RSP analysis has a logarithmic term.