Solutions IV

1 The equation is equidimensional, so try a solution $y = x^\alpha$. The polynomial gives $\alpha^2 - \alpha + \lambda = 0$. The two roots are $\alpha_{1,2} = \frac{1\pm\sqrt{1-4\lambda}}{2}$. When $\lambda = \frac{1}{4}$, the general solution is $y = A_1\sqrt{x} + B_1\sqrt{x}\log x$, which only has the trivial solution from the BCs. We can write the general solution as

$$y = Ax^{\frac{1+\sqrt{1-4\lambda}}{2}} + Ax^{\frac{1-\sqrt{1+4\lambda}}{2}}.$$

From the BCs, we obtain

$$A + B = 0, \qquad 2^{\frac{1+\sqrt{1-4\lambda}}{2}}A + 2^{\frac{1-\sqrt{1+4\lambda}}{2}}B = 0.$$

Non-trivial solutions are possible when

$$2^{\frac{1+\sqrt{1-4\lambda}}{2}} = 2^{\frac{1-\sqrt{1-4\lambda}}{2}},$$

which requires

$$2^{\sqrt{1-4\lambda}} = 1.$$

For $\lambda < 1/4$, there is no solution. For $\lambda > 1/4$, we have

$$1 = 2^{\sqrt{4\lambda-1}i} = \cos\left(\sqrt{4\lambda-1}\log 2\right) + \mathrm{i}\sin\left(\sqrt{4\lambda-1}\log 2\right),$$

which gives

$$\lambda_n = \frac{1 + (2n\pi/\log 2)^2}{4}.$$

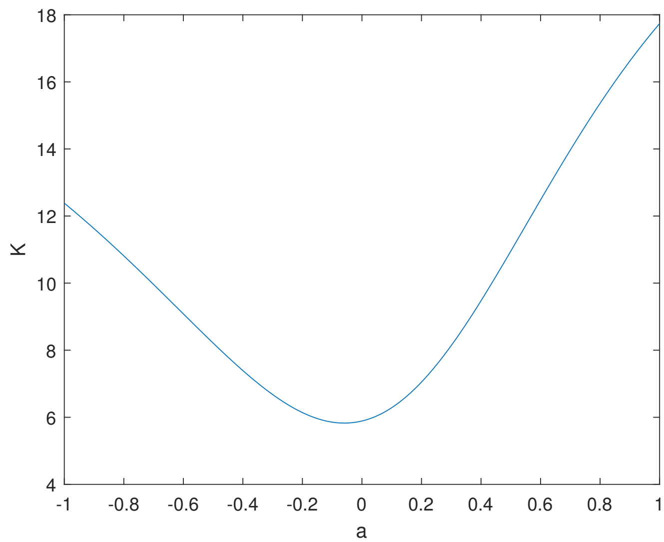
The two lowest eigenvalues are $\lambda_1 = 20.792$ and 82.419. For large n, λ_n behaves like $(n\pi/\log 2)^2$.

2 The Sturm–Liouville form gives $p = 1$, $q = 0$, $w = \mathrm{e}^x$. Try test function $y = \sin\pi x + a\sin 2\pi x$. The first derivative of y is $y' = \pi\cos\pi x + 2a\pi\cos 2\pi x$. We obtain

$$I = \int_0^1 (\pi\cos\pi x + 2a\pi\cos 2\pi x)^2\mathrm{d}x = \frac{1}{2}\pi^2 + 2a^2\pi^2.$$

$$J = \int_0^1 \mathrm{e}^x(\sin\pi x + a\sin 2\pi x)^2\mathrm{d}x = \frac{2(\mathrm{e}-1)\pi^2}{1+4\pi^2} - \frac{8a(1+\mathrm{e})\pi^2}{(1+\pi^2)(1+9\pi^2)} + \frac{8(\mathrm{e}-1)a^2\pi^2}{1+16\pi^2}.$$

The Rayleigh quotient is given by $K = I/J$, and its minimum is 5.827.



The exact solution can be obtained using the change of variable $t = 2\sqrt{\lambda}e^{x/2}$. Then the equation becomes

$$y_{tt} + \frac{1}{t}y_t + y = 0,$$

which is a Bessel equqtion with solution $y = AJ_0(t) + BY_0(t) = AJ_0(2\sqrt{\lambda}e^{x/2}) + BY_0(2\sqrt{\lambda}e^{x/2})$. The BCs give

$$AJ_0(2\sqrt{\lambda_n}) + BY_0(2\sqrt{\lambda_n}) = 0,$$
$$AJ_0(2\sqrt{\lambda_n}e^{1/2}) + BY_0(2\sqrt{\lambda_n}e^{1/2}) = 0.$$

The condition for non-trivial solutions is

$$J_0(2\sqrt{\lambda_n})Y_0(2\sqrt{\lambda_n}e^{1/2}) = Y_0(2\sqrt{\lambda_n})J_0(2\sqrt{\lambda_n}e^{1/2}).$$

The eigenvalues can be found numerically and $\lambda_1 \approx 5.82654627418$.

3 This is a Ricatti equation. We can spot the solution $y_1 = \sin x$, and then use the substitution $y = \sin x + u$ to get

$$u' = u \sin x + u^2$$

which is now Bernoulli with $p = 2$. Use $v = u^{1-p} = u^{-1}$ to get

$$-v' = v \sin x + 1.$$

Use the integrating factor $e^{\int^x \sin a \, da} = e^{-\cos x}$ to get

$$v(x) = e^{\cos x}\left(A - \int^x e^{-\cos a \, da} \, dx\right).$$

so the general solution is

$$y(x) = \sin x + \frac{\mathrm{e}^{-\cos x}}{A - \int^x \mathrm{e}^{-\cos a}\,\mathrm{d}a}.$$

4 Reexpress in terms of differentials:

$$x\mathrm{e}^{-y}\,\mathrm{d}x + (x^2+y^2)\,\mathrm{d}y = 0$$

with $A = x\mathrm{e}^{-y}$ and $B = x^2+y^2$. Then

$$\frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) = \frac{1}{x\mathrm{e}^{-y}}\left(2x + x\mathrm{e}^{-y}\right) = 2\mathrm{e}^{y} + 1 = f(y).$$

Hence inexact with integrating factor satisfying

$$\frac{\mathrm{d}\mu}{\mu} = (2\mathrm{e}^{y}+1)\,\mathrm{d}y,$$

which integrates to give

$$\mu = \exp\left(2\mathrm{e}^{y} + y\right).$$

The exact equation is

$$x\exp\left(2\mathrm{e}^{y}\right)\mathrm{d}x + (x^2+y^2)\exp\left(2\mathrm{e}^{y}+y\right)\mathrm{d}y = 0.$$

The resulting integral is

$$\frac{x^2}{2}\exp\left(2\mathrm{e}^{y}\right) + \int^y u^2\exp\left(2\mathrm{e}^{u}+u\right)\mathrm{d}u = C.$$

This can be solved explicitly for $x(y)$.

5 This is an equidimensional-in-x equation. Let $x = \mathrm{e}^t$ and find

$$y_{tt} - y_t + y_t + y^2 = y_{tt} + y^2 = 0.$$

Now this is autonomous and the first integral is automatic by multiplying by y_t:

$$\frac{1}{2}y_t^2 + \frac{1}{3}y^3 = A.$$

Now separate variables and get

$$x(y) = \exp\left(\int^y \frac{\mathrm{d}u}{[2(A-u^3/3)]^{1/2}} + B\right).$$

The solution $y(x)$ can be written in terms of the Weierstrass elliptic function ($\wp$).

6 This is an equidimensional-in-y equation. Let $y = e^u$ and find

$$u' - \frac{x}{1+u'} = 0.$$

This is a quadratic equation in u' with roots

$$u' = \frac{-1 \pm \sqrt{1+4x}}{2}.$$

This integrates up to

$$u = \frac{-x \pm (1+4x)^{3/2}/6}{2} + C.$$

The final answer is

$$y(x) = D \exp\left(-\frac{x}{2} \pm \frac{(1+4x)^{3/2}}{12}\right).$$