Solutions V

1 (a) Characteristic equation:

$$\frac{\mathrm{d}x}{y-2} = \frac{\mathrm{d}y}{x+1}.$$

Separate variables and obtain

$$p = \frac{x^2}{2} + x - \frac{y^2}{2} + 2y,$$

with $u(x,y) = f(p)$. The boundary condition gives $x = f(x^2/2 + x)$, so that $f(z) = \pm\sqrt{2z+1} - 1$ and

$$u(x,y) = \pm\sqrt{1 + x^2 + 2x - y^2 + 4y} - 1$$

with the $+$ sign for $p > -1$ and the $-$ sign for $p < -1$.
(b) Characteristic equation:

$$\frac{\mathrm{d}x}{\mathrm{e}^y} = \mathrm{d}y.$$

Find $p = x - \mathrm{e}^y$. The boundary condition gives $1 - x = f(x-1)$, so

$$u(x,y) = \mathrm{e}^y - x.$$

(c) Characteristic equation:

$$\frac{\mathrm{d}x}{\sin y} = \frac{\mathrm{d}y}{x}.$$

Separate variables and obtain

$$p = \frac{x^2}{2} + \cos y.$$

The boundary condition gives $x^2 = f(x^2/2 + 1)$, so

$$u(x,y) = x^2 + 2\cos y - 2.$$

(d) Characteristic equation:

$$\mathrm{d}x = \frac{\mathrm{d}y}{3x^2}.$$

Find $p = x^3 - y$. The boundary condition gives $1 = f(x^3 - x)$. This holds for all x and the argument of f takes all real values, so

$$u(x,y) = 1.$$

2 (a) A particular solution is x. The general solution is

$$u(x,y) = x + f\left(\frac{x^2}{2} + x - \frac{y^2}{2} + 2y\right).$$

The boundary condition gives $x = x + f(x^2/2 + x)$, so that $f = 0$ and $u(x,y) = x$.
(b) A particular multiplicative solution is $e^{y^2/2}$. The general solution is

$$u(x,y) = e^{y^2/2} f(x - e^y).$$

The boundary condition gives $1 - x = f(x-1)$ again, so

$$u(x,y) = (e^y - x)e^{y^2/2}.$$

(c) A particular solution is $-\cos y$. The general solution is

$$u(x,y) = -\cos y + f(x^2/2 + \cos y).$$

The boundary condition gives $x^2 = -1 + f(x^2/2 + 1)$, so so

$$u(x,y) = x^2 + \cos y - 1.$$

(d) Particular solutions are $x^3/3$ or $y/3$. The general solution is

$$u(x,y) = \frac{x^3}{3} + f(x^3 - y).$$

The boundary condition gives $1 = x^3/3 + f(x^3 - x)$. This is formally correct. However, Figure 1 shows that the function $f(z)$ is multivalued, which is problematic. It might be safest to say that the problem has no solution.

3 This is a Jacobian equation, as can be seen from

$$\frac{\partial}{\partial x}(2xy + \sin y) + \frac{\partial}{\partial y}(e^x - y^2) = 0.$$

Writing the original equation as $au_x + bu_y = 0$, the general solution is $u(x,y) = f(v)$, where $v_y = a$ and $v_x = -b$. This gives

$$v = xy^2 - e^x - \cos y.$$

The boundary condition leads to $x = f(-e^x - 1)$, so $f(v) = \log(-1 - v)$ and the solution is

$$u(x,y) = \ln(-1 + e^x - xy^2 + \cos y).$$

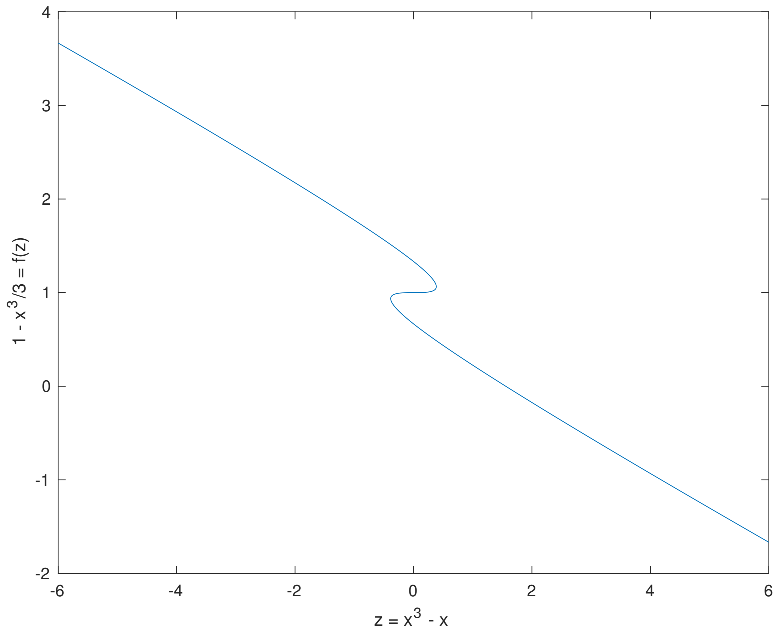

Figure 1: Multivalued function $f(x^3 - x)$.

4 For this problem, $A = 1$, $B = 1$, $C = -2$. This leads to $\lambda = 1, -1/2$. There are three simple choices for the particular solution: x^2, $2xy$ and $-y^2/2$. Take the last, so that the general solution is

$$u(x, y) = f(x + y) + g(x - y/2) - \frac{y^2}{2}.$$

The two boundary conditions give

$$f(x) + g(x) = 3x, \qquad f'(x) - \frac{1}{2}g'(x) = 2.$$

Differentiating the first expression and solving gives $f'(x) = 7/3$ and $g'(x) = 2/3$. The constant of integration is zero from the first boundary condition, so that

$$u(x, y) = \frac{7}{3}(x + y) + \frac{2}{3}\left(x - \frac{y}{2}\right) - \frac{y^2}{2} = 3x + 2y - \frac{y^2}{2}.$$

5 For this problem, $A = 1$, $B = -1$, $C = -1$. This leads to $\lambda_{\pm} = (-1 \pm \sqrt{5})/2$. The two boundary conditions give

$$f(x + \lambda_+ x) + g(x + \lambda_- x) = 3, \qquad f'(x + \lambda_+ x) + g'(x + \lambda_- x) = 1.$$

One can solve these equation to find the solution. A more efficient approach is to note from these boundary conditions that the solution must take the form $u(x, y) = ax + by + c$. The boundary conditions become $ax + bx + c = 3$ and $a = 1$ on $y = x$. Hence $a + b = 0$, $c = 3$ and $a = 1$. The final solution is

$$u(x, y) = x - y + 3.$$

6 For this problem, $A = 1$, $B = -2$, $C = 2$. This leads to $\lambda_\pm = (1 \pm \mathrm{i})/2$. The general solution is

$$u(x,y) = f\left(x + \frac{1+\mathrm{i}}{2}y\right) + g\left(x + \frac{1-\mathrm{i}}{2}y\right).$$