Solutions VI

Note that

$$\begin{aligned}
\int_0^{\pi} \theta \sin m\theta \, \mathrm{d}\theta &= \frac{(-1)^{m+1}\pi}{m}, \\
\int_0^{\pi} \theta(\pi - \theta) \sin m\theta \, \mathrm{d}\theta &= \begin{cases} 0 & \text{if } m \text{ is even,} \\ \dfrac{4}{m^3} & \text{if } m \text{ is odd,} \end{cases} \\
\int_0^{\pi} \theta^3 \sin m\theta \, \mathrm{d}\theta &= \frac{(-1)^{m+1}(\pi^2 m^2 - 6)\pi}{m^3}.
\end{aligned}$$

1 A particular integral is $u = xy^3/6$. Now solve Laplace's equation with boundary conditions $u = -ay^3/6$ on $x = a$ and $u = -xb^3/6$ on $y = b$. The former is as in class, with

$$u_1 = \sum_{m=1}^{\infty} A_m \sin \frac{m\pi y}{b} \sinh \frac{m\pi x}{b}.$$

Computing the coefficients in the Fourier series gives

$$A_m = -\frac{2}{b \sinh(m\pi a/b)} \int_0^b \frac{ay^3}{6} \sin \frac{m\pi y}{b} \, \mathrm{d}y = \frac{ab^3(-1)^m(\pi^2 m^2 - 6)}{3\pi^3 m^3 \sinh(m\pi a/b)}.$$

The latter gives

$$u_2 = \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{a} \sinh \frac{m\pi y}{a}.$$

Computing the coefficients in the Fourier series gives

$$B_m = -\frac{2}{a \sinh(m\pi b/a)} \int_0^a \frac{xb^3}{6} \sin \frac{m\pi x}{a} \, \mathrm{d}x = \frac{ab^3(-1)^m}{3m\pi \sinh(m\pi b/a)}.$$

The solution is

$$u(x,y) = u_1 + u_2 + \frac{xy^3}{6}.$$

2 The separation of variables is similar to Laplace's equation. We need trigonometric functions in y and the boundary conditions give

$$u = \sum_{m=1}^{\infty} A_m \cos \frac{m\pi y}{2b} \begin{cases} \sinh\left[(a-x)\sqrt{(\frac{m\pi}{2b})^2 - k^2}\right] & \text{for } k < m\pi/2b, \\ \sin\left[(a-x)\sqrt{k^2 - (\frac{m\pi}{2b})^2}\right] & \text{otherwise.} \end{cases}$$

The required Fourier coefficients for $k < m\pi/2b$ are

$$A_m = \frac{1}{b\sinh\left[2a\sqrt{(\frac{m\pi}{2b})^2 - k^2}\right]} \int_{-b}^{b} \cos\frac{m\pi y}{2b} = \frac{4}{m\pi\sinh\left[2a\sqrt{(\frac{m\pi}{2b})^2 - k^2}\right]} \sin\frac{m\pi}{2},$$

and

$$A_m = \frac{1}{b\sin\left[2a\sqrt{k^2 - (\frac{m\pi}{2b})^2}\right]} \int_{-b}^{b} \cos\frac{m\pi y}{2b} = \frac{4}{m\pi\sin\left[2a\sqrt{k^2 - (\frac{m\pi}{2b})^2}\right]} \sin\frac{m\pi}{2}$$

otherwise. Note that $\sin(m\pi/2) = (-1)^{m+1}$ for odd m and vanishes for even m.

3 Separate variables in the wave equation gives a Helmholtz equation. The time dependence is $\sin kct$ and $\cos kct$; the latter vanishes from the boundary condition $u = 0$ at $t = 0$. The azimuthal dependence leaves only $\sin 2\theta$. The radial equation leads to Bessel functions, and since the origin is in the domain only J_2 is allowed. Hence

$$u = \sum_{n=1}^{\infty} A_n J_2(k_n r) sin 2\theta \sin k_n ct,$$

where $k_n a = j_n$, with j_n the nth zero of $J_2(x)$. The initial condition gives

$$\mathrm{e}^{-r^2} = \sum_{m=1}^{\infty} A_n k_n c J_2(k_n r).$$

The orthogonality relation of Bessel functions in the interval $(0, 1)$ leads to

$$A_n = \frac{2}{a^2 k_n c J_3(j_n)^2} \int_0^a r J_2(k_n r) \mathrm{e}^{-r^2}\,\mathrm{d}r.$$

4 After separating variables as usual in plane polar coordinates, the appropriate expansion functions in θ are $\sin m\theta$, given the boundary conditions at $\theta = 0$ and $\theta = \pi$. Solutions with negative powers are unphysical at the origin, so

$$u = \sum_{m=1}^{\infty} A_m r^m \sin m\theta.$$

The boundary condition gives

$$A_m = \frac{2}{\pi a^m} \int_0^{\pi} \theta(\pi - \theta) \sin m\theta\,\mathrm{d}\theta = \frac{8}{\pi a^m m^3}$$

for m odd, and $A_m = 0$ for even m. The solution is

$$u = \sum_{m\,=\,1,\,\mathrm{odd}}^{\infty} \frac{8}{\pi a^m m^3} r^m \sin m\theta.$$

5 Separate variables in cylindrical polar coordinates. The typical separated solution that decays at infinity is

$$e^{-\kappa l^2 t} J_m(\lambda r) e^{im\phi} e^{-kz},$$

with $\lambda = \sqrt{k^2 + l^2}$, integer m and positive k. The problem is axisymmetric so we only retain $m = 0$. Given the initial condition we have $k = 1$. The boundary condition at $r = a$ gives $J_0(\lambda a) = 0$, and so $\lambda a = j_{0,p}$, the pth zero of J_0, i.e. $k^2 + l^2 = (j_{0,p}/a)^2$. Hence the superposed solution is

$$\sum_{p=1}^{\infty} J_0(j_{0,p} r/a) A_p(l) e^{-\kappa l^2 t} e^{-z}.$$

Now find the coefficients A_p from the initial condition using the Bessel orthogonality relation. We have

$$(a^2 - r^2) = \sum_{p=1}^{\infty} J_0(j_{0,p} r/a) A_p,$$

so

$$A_p(l) = \frac{2}{a^2 J_1(j_{0,p})^2} \int_0^a (a^2 - r^2) J_0(j_{0,p} r/a) r \, dr.$$

Note: this is the solution I wanted, but for it the boundary condition at $z = 0$ is not $T = 0$ but something more complicated which essentially shows the form of the solution. With the boundary condition as given, one has to work harder in z. Now the appropriate separated solutions are

$$e^{-\kappa k^2 t} J_m(\lambda r) e^{im\phi} \sin lz,$$

with $\lambda = \sqrt{k^2 - l^2}$ and integer m. The problem is axisymmetric so we only retain $m = 0$. The boundary condition at $r = a$ gives $J_0(\lambda a) = 0$, and so $\lambda a = j_{0,p}$, the pth zero of J_0, i.e. $k^2 - l^2 = (j_{0,p}/a)^2$. Hence the superposed solution is

$$\sum_{p=1}^{\infty} J_0(j_{0,p} r/a) \int_0^{\infty} A_p(l) e^{-\kappa k^2 t} \sin lz \, dl.$$

This is now an integral because l needs to take on all values as an inverse Fourier sine transform, with $k = \sqrt{l^2 + (j_{0,p}/a)^2}$. At $t = 0$, we have

$$e^{-z}(a^2 - r^2) = \sum_{p=1}^{\infty} J_0(j_{0,p} r/a) \int_0^{\infty} A_p(l) \sin lz \, dl.$$

This is beyond the scope of this course, but can be solved a double integral (which separates into two single integrals):

$$A_p(l) = \frac{4}{\pi a^2 [J_1(j_{0,p})]^2} \int_0^a (a^2 - r^2) J_0(j_{0,p} r/a) r \, dr \int_0^{\infty} e^{-z} \sin lz \, dl.$$

6 The time-dependence is harmonic, so we can find a solution $T = u\mathrm{e}^{-\mathrm{i}\omega t}$. Then

$$-\mathrm{i}\omega u = \kappa \nabla^2 u,$$

a Helmholtz equation with imaginary $k^2 = \mathrm{i}\omega/\kappa$. Separate variables in spherical polars. The θ and ϕ functions are the same as for the Laplace equation. Since the problem is axisymmetric, $m = 0$, while $\cos\theta$ corresponds to $P_1(\cos\theta)$, so that $l = 1$. Hence $u = g(r)\cos\theta$ with

$$g'' + \frac{2}{r}g' + \left(k^2 - \frac{2}{r^2}\right) g = 0.$$

The solution to this equation that is well behaved at the origin is $r^{-1/2}J_{3/2}(kr)$, which has a closed form expression in terms of trigonometric (hyperbolic functions since k is imaginary) and powers of r. The boundary condition gives $g(R) = 1$, so the solution is

$$T = \sqrt{\frac{R}{r}} \frac{J_{3/2}(kr)}{J_{3/2}(kR)} \mathrm{e}^{-\mathrm{i}\omega t} \cos\theta.$$