MMS Examples

1

$$\ddot{y} + \left[\frac{1}{1+(\epsilon t)^2} + 1\right]^2 y = 0.$$

Solution Use the change of independent variable as in class, with $\mu(T) = (1+T^2)^{-1} + 1$. Then

$$t_* = \frac{1}{\epsilon}\int_0^T \mu(S)\,\mathrm{d}S = \frac{\tan^{-1} T + T}{\epsilon}.$$

2

$$\ddot{y} + \epsilon t \mathrm{e}^{-t}\dot{y} + y = 0.$$

Solution Use a naive expansion. The leading-order solution is $y_0 = A\mathrm{e}^{-\mathrm{i}t} + c.c.$ At $O(\epsilon)$, find

$$y_{1tt} + y_1 = t\mathrm{e}^{-t}(\mathrm{i}A\mathrm{e}^{\mathrm{i}t} + c.c.),$$

with solution

$$y_1 = B\mathrm{e}^{\mathrm{i}t} + (Ct + D)A\mathrm{e}^{(\mathrm{i}-1)t} + c.c.,$$

where C and D can be found. There is no resonance.

3

$$\ddot{y} - \epsilon t \mathrm{e}^{-\epsilon t} y + y = 0.$$

Solution Use the change of independent variable as in class, with $\mu(T) = \sqrt{1 - T\mathrm{e}^{-T}}$. Then

$$t_* = \frac{1}{\epsilon}\int_0^T \sqrt{1 - S\mathrm{e}^{-S}}\,\mathrm{d}S.$$

4

$$\ddot{y} - \epsilon t \mathrm{e}^{\epsilon t}\dot{y} + y = 0.$$

Solution This is a WKB problem. See later.

5

$$\ddot{y} + \left[1 + \epsilon \frac{1+y^2}{1+\dot{y}^2}\right] y = 0.$$

Solution Skip the naive expansion. The leading-order solution is $x_0 = A(T)\mathrm{e}^{-\mathrm{i}t} + c.c.$ At $O(\epsilon)$, find

$$y_{1tt} + y_1 + 2y_{0Tt} + \frac{1+y_0^2}{1+y_{0t}^2} y_0 = 0.$$

The resonant term comes from $\mathrm{e}^{\mathrm{i}t}$, so multiplying the equation by $\mathrm{e}^{-\mathrm{i}t}$ and integrating over a fast period gives

$$2\mathrm{i}A_T + \oint \frac{1+(Az+A^*z^{-1})^2}{1+(\mathrm{i}Az-\mathrm{i}A^*z^{-1})^2}(Az+A^*z^{-1})\frac{\mathrm{d}z}{2\pi\mathrm{i}z} = 0,$$

using the change of variable $z = \mathrm{e}^{\mathrm{i}t}$ and noting that on the unit circle $\mathrm{e}^{\mathrm{i}t} = z^{-1}$. The result is

$$2\mathrm{i}A_T + F(A, A^*) = 0,$$

where $F(A, A^*)$ is given by evaluating the integral.