WKB Examples

1 The vertical normal modes in an ocean with stratification $N(z)$ are found from the eigenproblem

$$a'' + c^{-2}N^2(z)a = 0, \qquad a(0) = a(h) = 0.$$

The eigenvalue is c, the phase speed. Using WKB (LG might be a more accurate designation), solve this problem for general $N(z)$.

Solution From the lectures, we can identify E with c^{-2} and $N^2(z)$ with $Q(z)$, giving

$$y = CN^{-1/2}(z)\sin\left[c^{-1}\int_0^z N(u)\,\mathrm{d}u\right],$$

taking $N(z) > 0$ (which is physically correct). The eigenvalues are then given by

$$c = \frac{\int_0^h N(u)\,\mathrm{d}u}{n\pi}.$$

(There is a choice for the sign of c; it is conventional to take $c > 0$.) For orthonormal eigenfunctions, write

$$\varphi(z) = \frac{\int_0^x N(u)\,\mathrm{d}u}{\int_0^h N(u)\,\mathrm{d}u}$$

so that $y_n = C_n N^{-1/2}(z)\sin[n\varphi(z)]$. Then from the properties of trigonometric functions

$$\frac{\pi}{2}C_m C_n \delta_{mn} = C_m C_n \int_0^1 \sin m\pi\varphi \sin n\pi\varphi\,\mathrm{d}\varphi = \int_0^h y_m(z)y_n(z)N(z)\frac{\mathrm{d}\varphi}{\mathrm{d}z}\,\mathrm{d}z.$$

But $\mathrm{d}\varphi/\mathrm{d}z = N(z)/\int_0^h N(u)\,\mathrm{d}u$, so we take

$$C_n = \left(\frac{\pi}{2}\int_0^h N(u)\,\mathrm{d}u\right)^{-1/2}$$

and obtain

$$\int_0^h y_m(z)y_n(z)N^2(z)\,\mathrm{d}z = \delta_{mn}.$$

2 Find the large eigenvalues of the problem

$$y'' + E(1 - |x|)y = 0$$

where y decays at $\pm\infty$.

Solution The WKB solution that satisfies the connection formulas at $x = -1$ is

$$y \sim (1-|x|)^{-1/4} \cos\left(E^{1/2} \int_{-1}^{x} \sqrt{1-|u|}\, \mathrm{d}u - \frac{\pi}{4}\right).$$

Even solutions require

$$\cos\left(E^{1/2} \int_{-1}^{x} \sqrt{1-|u|}\, \mathrm{d}u - \frac{\pi}{4}\right) = \cos\left(E^{1/2} \int_{-1}^{-x} \sqrt{1-|u|}\, \mathrm{d}u - \frac{\pi}{4}\right).$$

Now $\cos A = \cos B$ if $A = B + 2n\pi$ or $A = -B + 2n\pi$. Changing variable in the second integral gives

$$E^{1/2} \int_{-1}^{x} \sqrt{1-|u|}\, \mathrm{d}u - \frac{\pi}{4} = \pm\left(E^{1/2} \int_{x}^{1} \sqrt{1-|u|}\, \mathrm{d}u - \frac{\pi}{4}\right) + 2n\pi.$$

The plus sign is too restrictive. The minus sign leads to

$$E^{1/2} \int_{-1}^{1} \sqrt{1-|u|}\, \mathrm{d}u = \frac{4}{3}E^{1/2} = \frac{\pi}{2} + 2n\pi.$$

For odd solutions, $\cos A = -\cos B$ if $A = B - \pi + 2n\pi$ or $A = -B - \pi + 2n\pi$. Changing variable in the second integral gives

$$E^{1/2} \int_{-1}^{x} [f(u)]^{1/2}\, \mathrm{d}u - \frac{\pi}{4} = \pm\left(E^{1/2} \int_{x}^{1} [f(u)]^{1/2}\, \mathrm{d}u - \frac{\pi}{4}\right) - \pi + 2n\pi.$$

The plus sign is too restrictive. The minus sign leads to

$$E^{1/2} \int_{-1}^{1} \sqrt{1-|u|}\, \mathrm{d}u = \frac{4}{3}E^{1/2} = -\frac{\pi}{2} + 2n\pi.$$

Hence the eigenvalues are

$$E \approx \frac{9\pi^2}{16}(n - \tfrac{1}{2})^2$$

for $n = 1, 2, \ldots$.

3 Find approximations to the large eigenvalues of the problem

$$y'' + EQ(x)y = 0, \qquad a_0 y'(0) + b_0 y(0) = 0, \qquad a_1 y'(1) + b_1 y(1) = 0,$$

where $a_0 \neq 0$, $a_1 \neq 0$ and $Q(x) > 0$ in the interval $(0, 1)$. Discuss the role of b_0 and b_1. Compare to the exact solution for $Q(x) = 1$.

Solution The L-G solution is

$$y = A[Q(x)]^{-1/4}\sin\left(E^{1/2}\int_0^x \sqrt{Q(u)}\,\mathrm{d}u\right) + B[Q(x)]^{-1/4}\cos\left(E^{1/2}\int_0^x \sqrt{Q(u)}\,\mathrm{d}u\right).$$

Differentiating e.g. the sine term gives

$$-\frac{1}{4}Q'(x)[Q(x)]^{-5/4}\sin\left(E^{1/2}\int_0^x \sqrt{Q(u)}\,\mathrm{d}u\right) + E^{1/2}[Q(x)]^{1/4}\cos\left(E^{1/2}\int_0^x \sqrt{Q(u)}\,\mathrm{d}u\right).$$

Since E is large, we can neglect the first term. Then in the two boundary conditions the b_0 and b_1 terms can also be neglected. The boundary condition at $x = 0$ gives $A = 0$ and the boundary condition at $x = 1$ gives

$$BQ(1)^{-1/4}\sin\left(E^{1/2}\int_0^1 \sqrt{Q(u)}\,\mathrm{d}u\right) = 0.$$

For a non-trivial solution, we obtain

$$E \sim \left(\frac{n\pi}{\int_0^1 \sqrt{Q(u)}\,\mathrm{d}u}\right)^2.$$

For the special case $Q = 1$, the exact solution is

$$y = A\sin(E^{1/2}x) + B\cos(E^{1/2}x).$$

The two boundary conditions can be written as a homogeneous matrix equation, and hence the exact eigenvalue condition is that the determinant

$$a_0E^{1/2}(-a_1E^{1/2}s + b_1c) - b_0(a_1E^{1/2}c + b_1s) = 0$$

vanish, with $s = \sin E^{1/2}$ and $c = \cos E^{1/2}$. The approximation above corresponds to keeping the $O(E)$ term; $a_0a_1s = 0$, so that $E \sim (n\pi)^2$. We can reduce the numbers of parameters from 4 to 2 by dividing by $a_0a_1 \neq 0$. Then the eigenvalue condition is

$$-Es + E^{1/2}(d_1 - d_0)c - d_0d_1s = 0.$$

where $d_0 = b_0/a_0$ and $d_1 = b_1/a_1$. This equation can be solve numerically starting with the guess $E \sim (n\pi)^2$.

4 Obtain a Liouville–Green type expansion for the fourth-order equation

$$y^{(4)} + a(x)y'' + b(x)y' + \lambda^2c(x)y = 0, \qquad \lambda \gg 1.$$

Find approximate eigenvalues for arbitrary $a(x)$, $b(x)$ and $c(x) > 0$ on the interval $(0, 1)$ with boundary conditions $y(0) = y'(0) = y(1) = y'(1)$. Why is there no loss of generality in not having a y''' term in the equation?

Solution Try the L–G ansatz $y = \mathrm{e}^{\lambda^\alpha \phi}$ where $\phi = \phi_0 + \lambda^\beta \phi_1 + \cdots$. The two largest terms when this is substituted into the governing equation give

$$\lambda^{4\alpha}(\phi_0')^4 + \lambda^2 c(x) + \cdots = 0.$$

Hence $\alpha = 1/2$ and $\phi_0' = \{\mathrm{e}^{\pm \mathrm{i}\pi/4}, \mathrm{e}^{\pm 3\mathrm{i}\pi/4}\}[c(x)]^{1/4}$. The next terms are

$$\lambda^{2-\beta} 4(\phi_0')^3 \phi_1' + \lambda^{3/2} 6(\phi_0')^2 \phi_0'' + \cdots = 0.$$

This gives $\beta = -1/2$ and $\phi_1 \sim [c(x)]^{-3/8}$. Hence the LG expansion is

$$y = [c(x)]^{-3/8} \exp\left[\lambda^{-1/2}\{\mathrm{e}^{\pm \mathrm{i}/4}, \mathrm{e}^{\pm 3\mathrm{i}/4}\} \int_0^x [c(u)]^{1/4}\, \mathrm{d}u\right].$$

For the rest of the question, it is most convenient to consider the solution in the form

$$\begin{aligned} y \;=\; & \alpha[c(x)]^{-3/8} \cosh[\lambda^{1/2}\theta] \cos[\lambda^{1/2}\theta] + \beta[c(x)]^{-3/8} \cosh[\lambda^{1/2}\theta] \sin[\lambda^{1/2}\theta] \\ & + \gamma[c(x)]^{-3/8} \sinh[\lambda^{1/2}\theta] \cos[\lambda^{1/2}\theta] + \delta[c(x)]^{-3/8} \sinh[\lambda^{1/2}\theta] \sin[\lambda^{1/2}\theta], \end{aligned}$$

where $\theta(x) = 2^{-1/2} \int_0^x [c(u)]^{1/4}\, \mathrm{d}u$. As in **1** above, ignore the prefactor when differentiating. Then writing $C = \cosh[\lambda^{1/2}\theta(1)]$, $S = \sinh[\lambda^{1/2}\theta(1)]$, $c = \cos[\lambda^{1/2}\theta(1)]$, $s = \sin[\lambda^{1/2}\theta(1)]$, the boundary conditions become

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ Cc & Cs & Sc & Ss \\ Sc - Cs & Ss + Cc & Cc - Ss & Cs + Sc \end{vmatrix} = S^2 - s^2 = 0.$$

The only real root of this relation is $\lambda = 0$ which is not interesting.
However, one also has to consider the possibility that in fact $\lambda^2 = -\mu^2 < 0$. Then the solution can be written

$$\begin{aligned} y = \; & \alpha[c(x)]^{-3/8} \cosh[\mu^{1/2}\chi] + \beta[c(x)]^{-3/8} \sinh[\mu^{1/2}\chi] \\ & + \gamma[c(x)]^{-3/8} \cos[\mu^{1/2}\chi] + \delta[c(x)]^{-3/8} \sin[\mu^{1/2}\chi], \end{aligned}$$

with $\chi(x) = \int_0^x [c(u)]^{1/4}\, \mathrm{d}u$. The boundary conditions at the origin leads to $\alpha + \beta = \gamma + \delta = 0$ and the other boundary conditions give

$$\begin{vmatrix} C - c & S - s \\ S + s & C - c \end{vmatrix} = 2(1 - Cc) = 0.$$

Hence we need to solve the equation $\cosh z \cos z = 1$. Then $\mu^{1/2}\chi(1) = z$. Graphically this can be seen to have an infinity of positive solutions z_n, so we find

$$\mu = \left(\frac{z_n}{\int_0^1 [c(u)]^{1/4}\, \mathrm{d}u} \right)^2.$$

Ignoring the root at the origin, the numerical values of the z_n are 4.730040744862704, 7.853204624095838, 10.995607838001671, ... For large n, $z_n \sim (n + 1/2)\pi$. If there were a $z(x)y'''$ term in the equation, the change of variable $y = wg$ would lead to two terms in w''': $w'''(zg + 4g')$. Setting this to zero, which gives $g = \exp[-(1/4)\int z]$ removes the w''' term. Hence one can transform remove the third derivative term from a fourth-order linear ODE.