Integrity statement: I certify that I have completed this test on my own under the conditions listed below in the time allotted. Signed ____________

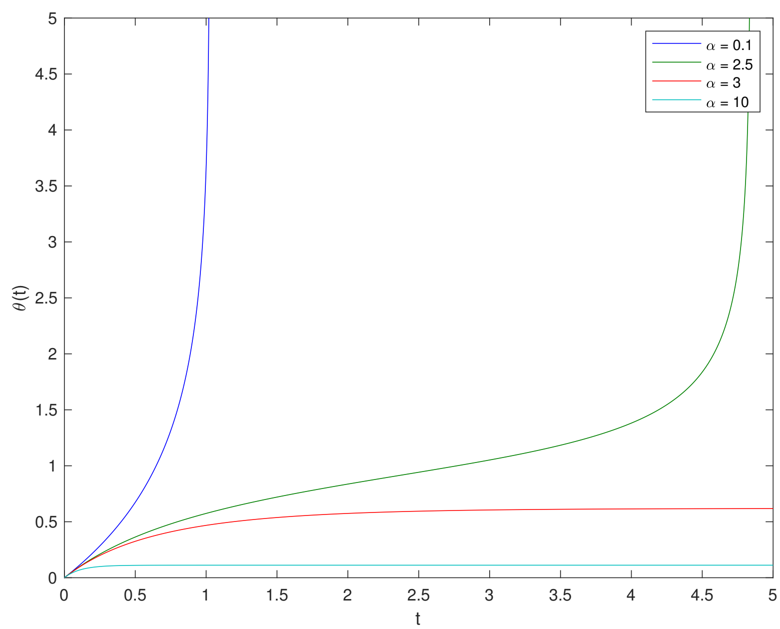
# Final

This is a 3-hour open-note exam (no books). Answer all six questions. The equations can be answered independently, but some relate back to **1**.

The theory of thermal explosions was developed by Russian scientists Frank-Kamenetskii and Zel'dovich. A model problem due to Semyonov describing time-dependent ignition in vessels filled with a reactant mixture is

$$\dot{\theta} = e^{\theta} - \alpha\theta, \qquad \theta(0) = 0,$$

where $\theta$ denotes temperature and $\alpha > 0$ is the controlling parameter (the inverse of the relevant Damköhler number). The figure below shows numerical calculations of the solution for four values of $\alpha$.




**1. Dynamical system analysis** Carry out a phase line analysis. Show that fixed points $\theta_0$ exist for $\alpha > \alpha_*$. What is $\alpha_*$? Discuss the stability of the fixed points. Which fixed point is relevant, given the initial condition? Sketch the bifurcation diagram in the $(\alpha, \theta)$ plane. For $\alpha \gg 1$, find two terms in perturbative solutions for both values of $\theta_0$.

**2. Ignition time** Show that the solution can develop a finite-time singularity at $t = t_*$ with

$$t_* = \int_0^\infty \frac{\mathrm{d}u}{\mathrm{e}^u - \alpha u}.$$

For what values of $\alpha$ does this happen? Obtain a series expansion of $t_*$ in $\alpha$. Bonus: what is its radius of convergence? [Reminder:

$$\Gamma(x) = \int_0^\infty s^{x-1}\mathrm{e}^{-s}\,\mathrm{d}s$$

for Re $x > 0$, $\Gamma(x+1) = x\Gamma(x)$, $n! = \Gamma(n+1)$, and

$$\lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n = \mathrm{e}^y.$$

for all $y$.]

**3 Slow ignition** This is the case when $\alpha = \alpha_* - \epsilon$ for $0 \ll \epsilon \ll 1$. Show that the denominator of the integrand in $t_*$ vanishes at $u = 1$ when $\epsilon = 0$ and is locally a quadratic near $u = 1$. Hence argue that, to leading order, $t_*$ is approximated by a local integral in $v$ where $u = 1 + \epsilon^{1/2}v$. Deduce that $t_* = \pi(2/\mathrm{e})^{1/2}\epsilon^{-1/2} + \cdots$.

**4 Fast equilibration** For $\alpha \gg 1$, obtain two terms in a regular perturbation series using the rescaling $\Theta = \alpha\theta$ and $T = \alpha t$. Now argue that the expansion is not uniformly valid in time and obtain one term using MMS, where the slow time is the original variable $t$. Why is this irrelevant in practice?

**5 Fast ignition** For $\alpha \ll 1$, obtain the first two terms in a series expansion for $\theta(t)$ in $\alpha$. The expansion becomes disordered in a region around $t = 1$ of width $\alpha$ (neglecting logarithmic corrections). Obtain the governing equation for a local solution of the form $\Theta = \log\alpha^{-1} + \Theta_0(T) + \cdots$, where $t = 1 - \alpha T$. Show that $\Theta_0(T) = \log[1/(T+c)]$ and match to the outer solution to find $c$. [Note: use van Dyke with $(1,0)$ in the outer solution and $(0,1)$ in the inner, keeping track of logarithmic terms.]

**6 Frank-Kamenetskii model** This replaces the term in $\alpha$ by a diffusion term with respect to a spatial variable $x$, so that the equation becomes

$$\frac{\partial\theta}{\partial t} = \mathrm{e}^\theta + \frac{1}{\delta}\frac{\partial^2\theta}{\partial x^2}$$

with $\delta > 0$. The boundary conditions in space are $\theta_x = 0$ at $x = 0$ and $\theta = 0$ at $x = 1$. For the steady state, the main quantity of interest is $\theta_m = \theta(0)$. For $\delta \ll 1$, show that there is a solution with $\theta_m = \frac{1}{2}\delta + O(\delta^2)$. [Note: this is a regular perturbation problem.]