Solution to Final

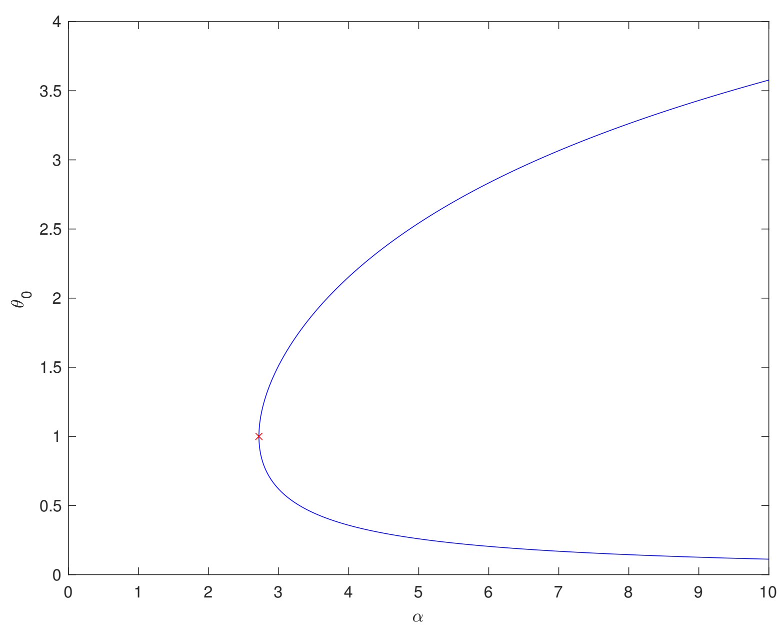
1. Dynamical system analysis Phase line analysis (2): we have $\dot{\theta} = f(\theta)$, with $f(\theta) = e^{\theta} - \alpha\theta$. Fixed points are at $\alpha = \theta^{-1}e^{\theta}$. This has a minimum at $\theta_* = 1$ and $\alpha_* = e$. Hence there are no fixed points with $\alpha > e$ and two fixed points when $\alpha < e$ (2). There is a saddle-node bifurcation, at which the fixed point is semi-stable. See the bifurcation diagram below (2). The smaller fixed point is stable ($f' < 0$) and the larger one is unstable ($f' > 0$) (1). The smaller one is the relevant one, since the initial condition is to its left and $f' < 0$ to its left (1). For $\alpha \gg 1$, the equation $f(\theta) = 0$ has a small root and a small root. We try $\theta_0 = \alpha^{-1}x_1 + \alpha^{-2}x_2$. Then

$$0 = 1 + \alpha^{-1}x_1 - x_1 - \alpha^{-1}x_2 + O(\alpha^{-2}).$$

Hence $\theta_0 = \alpha^{-1} + \alpha^{-2} + \cdots$ (1). For the large root, we expect the exponential to dominate, so re-express the equation in terms of logarithms and try the iteration

$$\theta_{n+1} = \log(\alpha\theta_n), \qquad \theta_0 = \log\alpha = L_1,$$

with the usual L_n notation. Then $\theta_1 = \log(\alpha L_1) = L_1 + L_2$ and $\theta_1 = \log[\alpha(L_1 + L_2)] = L_1 + L_2 + \log(1 + L_2/L_1)$. Hence a two-term expansion is $\theta_0 = \log\alpha + \log\log\alpha + \cdots$ (1).



2. Ignition time Separating variables gives the exact solution

$$t = \int_0^\theta \frac{\mathrm{d}u}{\mathrm{e}^u - \alpha u}.$$

The integral on the right exists if $\alpha < \alpha_*$, corresponding to the case when the denominator does not vanish (2). In that case, we can take $\theta \to \infty$ and obtain a well-defined integral on the left-hand side: this is the singularity time t_* (3). Now expand the integrand in α (3):

$$t_* = \int_0^\infty \mathrm{e}^{-u} \sum_{n=0}^\infty (\alpha u)^n \mathrm{e}^{-nu}\, \mathrm{d}u = \sum_{n=0}^\infty \alpha^n \int_0^\infty u^n \mathrm{e}^{-(n+1)u}\, \mathrm{d}u,$$

interchanging sum and integral. Now the change of variable $s = (n+1)u$ gives (2)

$$t_* = \sum_{n=0}^\infty \alpha^n (n+1)^{-(n+1)} \int_0^\infty s^n \mathrm{e}^{-s}\, \mathrm{d}u = \sum_{n=0}^\infty \alpha^n n! (n+1)^{-(n+1)}.$$

Bonus: if the interchange is permissible, we expect to get a converging series rather than an asymptotic expansion. We examine the radius of convergence: the relevant ratio is

$$r = \frac{\alpha^n n! n^n}{\alpha^{n-1}(n-1)!(n+1)^{n+1}} = \alpha \frac{n}{n+1}\left(1 + \frac{1}{n}\right)^{-n} \to \alpha \mathrm{e}^{-1}$$

for large n. The series converges if in the limit $|r| < 1$. Hence the radius of convergence is exactly $\mathrm{e} = \alpha_*$, so the series always converges to t_* when t_* exists. One can also obtain this result by requiring that the binomial expansion converge, which it does if $\alpha u \mathrm{e}^{-u} < 1$ for all u. The minimum value of this quantity in the range of integration is $\alpha \mathrm{e}^{-1}$ at $u = 1$, so the radius of convergence is $\alpha = \mathrm{e}$ (Bonus 5).

3 Slow ignition The denominator is $d(u) = \mathrm{e}^u - \alpha u$. When $\epsilon = 0$, we recover the critical point with $d(1) = d'(1) = 0$, so that $d(u)$ is locally quadratic (3). When $\epsilon > 0$, we obtain $d(u) = O((u-1)^2) + \epsilon u$, so the appropriate variable to have these terms balance is v with $u = 1 + \epsilon^{1/2} v$. Now D&C (2). There are two global integrals and one local integral. The local integral is

$$\begin{aligned}
I_L &= \int_{-\delta/\epsilon^{1/2}}^{\delta/\epsilon^{1/2}} \frac{\epsilon^{1/2}\,\mathrm{d}v}{\mathrm{e}^{1+\epsilon^{1/2}v} - (\mathrm{e}-\epsilon)(1+\epsilon^{1/2}v)} = \int_{-\delta/\epsilon^{1/2}}^{\delta/\epsilon^{1/2}} \frac{\epsilon^{1/2}\,\mathrm{d}v}{\mathrm{e}\epsilon v^2/2 + \epsilon + O(\epsilon^{3/2}v^3, \epsilon^{3/2}v)} \\
&= \frac{1}{\epsilon^{1/2}} \int_{-\delta/\epsilon^{1/2}}^{\delta/\epsilon^{1/2}} \frac{\mathrm{d}v}{\mathrm{e}v^2/2 + 1}[1 + O(\epsilon^{1/2}v^{1/2}, \epsilon^{1/2}v)] \\
&= \left(\frac{2}{\mathrm{e}\epsilon}\right)^{1/2} \left[\tan^{-1}\frac{\mathrm{e}v}{\sqrt{2}}\right]_{-\delta/\epsilon^{1/2}}^{\delta/\epsilon^{1/2}} + O(\delta^{3/2}) = \left(\frac{2}{\mathrm{e}\epsilon}\right)^{1/2} \pi + O\left(\frac{\epsilon^{1/2}}{\delta}, \delta^{3/2}\right),
\end{aligned}$$

with $\delta \ll 1$ and $\delta \gg \epsilon^{1/2}$ (5). The global integrals together give

$$I_G = \left(\int_0^{1-\delta} + \int_{1+\delta}^\infty\right) \frac{\mathrm{d}u}{\mathrm{e}^u - \mathrm{e}u}\left(1 + \frac{\epsilon u}{\mathrm{e}^u - \mathrm{e}u}\right)^{-1},$$

which appears to be $O(1)$, which can be shown. This is the size of the correction term.

4 Fast equilibration The rescaled problem is

$$\Theta_T = e^{\alpha^{-1}\Theta} - \Theta, \qquad \Theta(0) = 0.$$

Now try the expansion $\Theta = \Theta_0(T) + \alpha^{-1}\Theta_1(T) + \cdots$, and go through the orders. At $O(1)$,

$$\Theta_{0T} = 1 - \Theta_0, \qquad \Theta_0(0) = 0,$$

with solution $\Theta_0(T) = 1 - e^{-T}$ (3). At $O(\alpha^{-1})$,

$$\Theta_{1T} = \Theta_0 - \Theta_1 = 1 - e^{-T} - \Theta_1, \qquad \Theta_1(0) = 0,$$

with solution $\Theta_1(T) = 1 - (1+T)e^{-T}$ (3). The Te^{-T} term is not uniformly valid, so we try MMS (1). The long time is now the original t. We find $\Theta_0(T,t) = 1 + A(t)e^{-T}$ with $A(0) = -1$, while the $O(\alpha^{-1})$ equation becomes

$$\Theta_{1T} + A_t e^{-T} = 1 + Ae^{-T} - \Theta_1.$$

Removing the terms in e^{-T} gives $A_t = A$. This has solution $A(t) = -e^t$, so the MMS solution is $\Theta = 1 - e^{(-1+\epsilon)t} + \cdots$ (2). In practice, the non-uniformity is irrelevant since the non-uniform terms decay (1).

5 Fast ignition Write $\theta = \theta_0 + \alpha\theta_1 + \cdots$, and expand the governing equation. At $O(1)$,

$$\dot{\theta}_0 = e^{\theta_0}, \qquad \theta_0(0) = 0,$$

with solution $\theta_0(t) = \log\left[1/(1-t)\right]$ (2). At $O(\alpha)$, we find

$$\dot{\theta}_1 = e^{\theta_0}\theta_1 - \theta_0 = \frac{\theta_1}{1-t} - \log\frac{1}{1-t}, \qquad \theta_1(0) = 0,$$

which can be solved to give (3)

$$\theta_1(t) = \frac{t^2/4 - t/2}{1-t} + \frac{1}{2}(1-t)\log\frac{1}{1-t}.$$

We see that the expansion becomes disordered close to $t = 1$. Substitute $\theta = \log\alpha^{-1} + \Theta_0(T) + \cdots$, with $t = 1 - \alpha T$, into the governing equations to obtain the leading-order equation (1)

$$\Theta_{0T} = -e^{\Theta_0},$$

with solution $\Theta_0(T) = \log\left[1/(T+c)\right]$ (2). Set up Van Dyke:

$$\theta^{(1,\cdot)} = \log\frac{1}{\alpha T} + \alpha\left(\frac{(1-\alpha T)t^2/4 - (1-\alpha T)/2}{\alpha T} + \frac{1}{2}\alpha T\log\frac{1}{\alpha T}\right)$$

and

$$\Theta^{(0,\cdot)} = \log\alpha^{-1} + \log\frac{\alpha}{1-t+\alpha c} = \log\frac{1}{1-t} - \log\left(1 + \frac{\alpha c}{1-t}\right).$$

Hence (2)

$$\theta^{(1,0)} = \log\frac{1}{\alpha T} - \frac{1}{4T} = \Theta^{(0,1)} = \log\frac{1}{1-t} - \frac{\alpha c}{1-t},$$

and $c = 1/4$.

6 Frank-Kamenetskii model The result suggests that the expansion starts at $O(\delta)$, but we can start at $O(1)$ anyway. Write $\theta = \theta_0 + \delta\theta_1 + \cdots$. At $O(1)$,

$$\theta_{0xx} = 0, \qquad \theta_{0x}(0) = 0, \qquad \theta_0(1) = 0,$$

with solution $\theta_0(x) = 0$ (4). At $O(\delta)$, we find

$$\theta_{1xx} = -1, \qquad \theta_{1x}(0) = 0, \qquad \theta_1(1) = 0,$$

with solution $\theta_1(x) = -\frac{1}{2}(x^2 - 1)$ (4) and in particular $\theta_1(0) = \frac{1}{2}$. This is a regular perturbation problem, so the next term will be at O(δ^2). Hence $\theta_m = \frac{1}{2}\delta + O(\delta^2)$ (2).