Homework I

Due Jan 17, 2019.

1 In class we showed that the pendulum equation

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

can be integrated to give the conversation law

$$\frac{1}{2}\dot{\theta}^2 + \frac{g}{l}(1 - \cos\theta) = E$$

with $E \geq 0$. Hence obtain the period of a pendulum as an integral in the form

$$T = 4\int_0^{\theta_*} \frac{\mathrm{d}\theta}{\sqrt{F(\theta)}},$$

where you should determine the function $F(\theta)$ and give an expression for θ_*. Obtain an approximation to the integral for small-amplitude oscillations, defining "small amplitude" carefully in terms of a non-dimensional parameter. Bonus: compute the integral numerically, either directly or by transforming it into an elliptic integral, as a function of the non-dimensional parameter.

2 Semi-stable points on the phase line have $f(x_*) = f'(x_*) = 0$ (and possible vanishing higher derivatives as well). Interpret this graphically. Examine the phase line for the equation

$$\dot{x} = x^3 - x^2.$$

Discuss the stability of the fixed points. Show that the fixed point at the origin is semi-stable. Solve for its local dynamics by neglecting the higher-order term, and discuss the behavior for large t. Now solve the problem exactly. What happens to the fixed points if $f(x)$ becomes $x^3 - x^2 - \delta$, where δ is a small positive number (a graphical argument will suffice)? Hence explain why semi-stable points are not "structurally stable."

3 Similar ideas to the phase line apply to the dynamics of maps, where we write $x_{n+1} = f(x_n)$. For a general map, interpret graphically the condition to have a fixed point $x_* = f(x_*)$. Linearize about a fixed point and show that the fixed point x_* is unstable (resp. unstable) if $|f'(x_*)| < 1$ (resp. $|f'(x_*)| > 1$). Interpret this condition graphically. One of the most famous maps is the logistic map, given by

$$x_{n+1} = 4rx_n(1 - x_n),$$

where we are interested in $0 \leq x \leq 1$. What is the range of r for which the map takes the interval $(0, 1)$ to itself? Find the fixed points of the map as a function of r. Discuss fixed points of the iterated map $x_{n+2} = x_n$. Bonus: produce a diagram of the attractor of the map as a function of r, say between 2.4 and 4, by iterating the map from a lot of initial conditions and plotting the x-values after discarding a number of early iterates. What do you see? [There are lots of sources of further information about this that you can find.]

4 Analyze the fixed points of the system

$$\dot{x} = (x - y)(1 - x - y)/3, \qquad \dot{y} = x(2 - y).$$

Sketch the phase plane.