Homework VI

Due March 12, 2020.

1 Calculate two terms in an expansion in $0 < \epsilon \ll 1$ for the problem

$$f'' - \epsilon \frac{ff'}{x^m} = 0, \qquad f(0) = 1, \quad f(1) = 0$$

with integer m. For $m \leq 0$, show that this is a regular perturbation problem. For $m = 2$, use an expansion of the form $f_0(x) + \epsilon \log \epsilon^{-1} f_1(x) + \epsilon f_2(x) + \cdots$ in the outer region, and $1 + \epsilon \log \epsilon^{-1} F_1(X) + \epsilon F_2(X)$ in an inner region with $x = \epsilon X$. Match. Discuss what happens for $m = 1$ (you do not need to calculate higher-order terms). Now solve the equation exactly for $m = 0$ and $m = 1$. Does this give insight into the asymptotic results? [You may use the following result:

$$\int_0^X \exp\left(\frac{u^{1-m}}{1-m}\right) \mathrm{d}u = \begin{cases} X + I_m + O(X^{2-m}) & \text{for } m > 2, \\ X - \log X + \gamma - 1 + O(X^{-1}) & \text{for } m = 2, \end{cases}$$

as $X \to \infty$, where $\gamma = 0.577\ldots$ is the Euler–Mascheroni constant.]

2 Compute two terms in the expansion of the integral

$$\int_0^\infty \frac{\mathrm{d}x}{(\epsilon + x)(1 + \epsilon x)}$$

for small positive ϵ. Compare with the exact solution.

3 Show that for large positive x

$$\int_{\pi/2}^\infty \mathrm{e}^{x(\cos^2 t - 1)} \frac{\mathrm{d}t}{t^2} \sim \frac{1}{6}\sqrt{\frac{\pi}{x}}.$$

[Hint: $\sum_{n=1}^\infty n^{-2} = \pi^2/6$.]

4 Compute the first term in the asymptotic expansion of

$$\int_0^\infty \cos\left[x(at^2 - t)\right] \cos\left[x(bt^2 + t)\right] \mathrm{d}t$$

for large positive x and $a > b > 0$. What happens as $a \to b$?