Midterm Solution

1 The fixed points are at $(-1,0)$, $(1,0)$ and $(2,3)$. The Jacobian matrix is

$$J = \begin{pmatrix} 2x & -1 \\ y & x-2 \end{pmatrix}.$$

For $(-1,0)$,

$$J = \begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix},$$

with eigenvalues/vectors -2: $(1,0)^T$ and -3: $(1,1)^T$. This is stable node with trajectories tangential to the first eigenvector. For $(1,0)$,

$$J = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix},$$

with eigenvalues/vectors -1: $(1,3)^T$ and 2: $(1,0)^T$. This is a saddle. For $(2,3)$,

$$J = \begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix},$$

with eigenvalues/vectors 1: $(1,3)^T$ and 3: $(1,1)^T$. This is an unstable node with trajectories tangential to the first eigenvector. Figure 1 shows the phase plane.

2 Infinity is an ISP, as can be checked. Write $w = \mathrm{e}^S$, giving

$$(1-\zeta^2)(S'' + S'^2) - \zeta S' + a + 2q - 4q\zeta^2 = 0.$$

Assume as usual that $S'' \ll S'^2$, and also that $\zeta S' \gg 1$, which does not seem very restrictive. Then the dominant balance is

$$S'^2 \sim -4q.$$

We find $S \sim \pm 2\mathrm{i}q^{1/2}\zeta$, which satisfies the assumptions above. Now write $S = \pm 2\mathrm{i}q^{1/2}\zeta + T$, where $T \ll \zeta$. This leads to

$$(1-\zeta^2)(T'' - 4q \pm 4\mathrm{i}q^{1/2}T' + T'^2) - \zeta(\pm 2\mathrm{i}q^{1/2} + T') + a + 2q - 4q\zeta^2 = 0.$$

The dominant balance now comes from the two terms $\mp 4\mathrm{i}q^{1/2}T'\zeta^2$ and $\mp 2\mathrm{i}q^{1/2}\zeta$, so that

$$T \sim -\frac{1}{2}\log\zeta.$$

This is enough to obtain the controlling behavior as

$$w = \zeta^{-1/2}\mathrm{e}^{\pm 2\mathrm{i}q^{1/2}\zeta}.$$

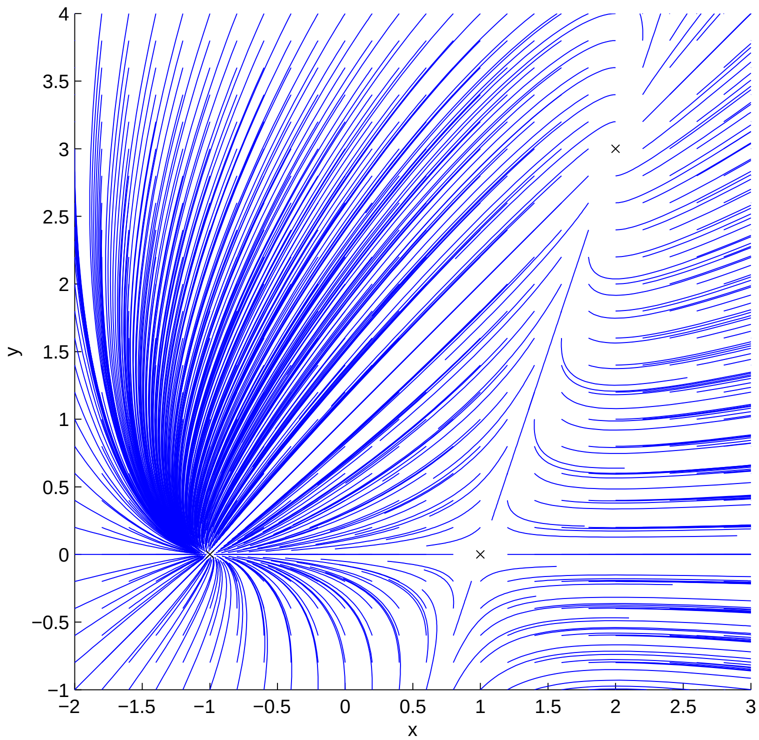

Figure 1: Phase plane.

3 *(a)* The distinguished scalings are 1 and ϵ^{-1}. The leading-order behavior for the $O(1)$ solution is the double root -1, so we expect the possibility of a different scaling for the expansion. Try $x = -1 + \delta_1 x_1 + \cdots$. Then

$$\epsilon(-1+\cdots) + 1 - 2\delta_1 x_1 + \delta_1^2 x_1^2 + \cdots + 2(-1 + \delta_1 x_1 + \cdots) + \epsilon(-1+\cdots) + 1 = 0.$$

The leading order balance after cancellation is $-2\epsilon + \delta_1^2 x_1^2 = 0$, so

$$x = -1 \pm \sqrt{2}\epsilon^{1/2} + \cdots.$$

The $O(\epsilon^{-1})$ root is of the form $x = \epsilon^{-1} x_{-1} + x_0 + \cdots$, where

$$x_{-1}^3 + x_{-1}^2 = 0, \qquad (3x_{-1}^2 + 2x_{-1})x_0 + 2x_{-1} = 0.$$

We ignore the double root $x_{-1} = 0$, which corresponds to the previous scaling and find

$$x = -\epsilon^{-1} + 2 + \cdots.$$

(b) Now $x = -1$ is an exact solution. A look at the preceding equations for the $O(1)$ root shows that due to cancellation, the correction term is $O(\epsilon)$. Algebra gives

$$x = -1 - 2\epsilon + \cdots.$$

The equations giving the first two terms of the singular root are not affected by the change in the equation, so again

$$x = -\epsilon^{-1} + 2 + \cdots.$$

4 The native expansion has a secular term at first order of the form te^{it}. Hence there is a slow time scale $T \equiv \epsilon t$ and we use MMS to obtain a solution uniformly valid for $T = \text{ord}(1)$. The fast time is $\tau \equiv t$ the usual process gives is $y_0 = A(T)e^{i\tau} + c.c.$, so that the first-order equation is

$$\frac{\partial^2 y_1}{\partial \tau^2} + y_1 = -2\frac{\partial^2 y_0}{\partial \tau \partial T} - \left[\left(\frac{\partial y_0}{\partial \tau}\right)^2 - 1\right]\frac{\partial y_0}{\partial \tau}.$$

The right-hand side is

$$-2iA_T e^{i\tau} + 2iA_T^* e^{-i\tau} - (-A^2 e^{2i\tau} - A^{*2} e^{-2i\tau} + 2AA^* - 1)(iAe^{i\tau} - iA^* e^{-i\tau}).$$

The secularity condition may hence be written as

$$2A_T = (1 - 3|A|^2)A.$$

Writing $A = Re^{i\Theta}$ gives

$$2R_T = (1 - 3R^2)R, \qquad R\Theta_t = 0.$$

The fixed point of the R-equation is $R^2 = 1/3$. It is stable and the solution will ultimately tend to this amplitude, provided the initial condition is not $y(0) = \dot{y}(0) = 0$. Hence the large-time behavior of the oscillator is

$$y \sim \frac{1}{\sqrt{3}} e^{i(\tau + \theta_0)} + c.c.,$$

which is an oscillation with peak-to-peak displacement $4/\sqrt{3}$.